On The Complexity of Counter Reachability Games

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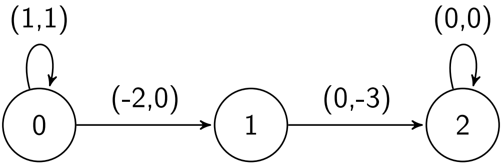
What is a counter system?

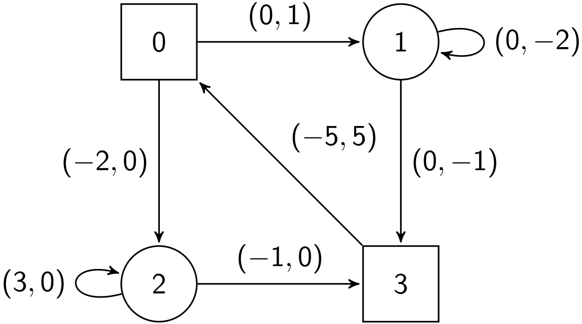


- d-dimensional **counter system** (CS): (Q, E).
- Example: $d = 2$, $Q = \{0, 1, 2\}$, $E = \{(0, (1, 1), 0),\ (0, (-2, 0), 1),\ (1, (0, -3), 2),\ (2, (0, 0), 2)\}$.
- **Configuration**: $c_i = (q_i, (v_1^i, \ldots, v_d^i)) \in Q \times \mathbb{Z}^d$.
- **Run** example:
  $(0, (1, 0)) \to (0, (2, 1)) \to (1, (0, 1)) \to (2, (0, -2)) \to \ldots$
  finite or not.

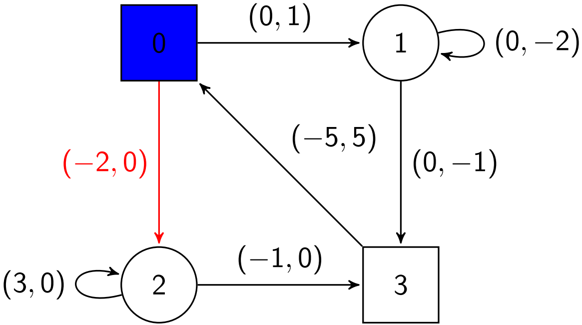
What is a counter system?

- $\mathbb{Z}$ **semantics**: counters are relative integers, no restriction on transitions.
- **Vector Addition System with States** (VASS): CS with an additional property: in each run, a transition is disabled if it would lead to a configuration with a negative counter value.
- **Non-blocking VASS semantics** (NBVASS): unlike for VASS, every transition is always enabled, negative counter values are immediately replaced by zero.

What is a counter reachability game?

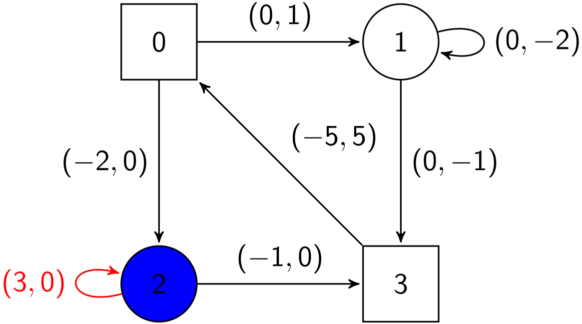


What is a counter reachability game?



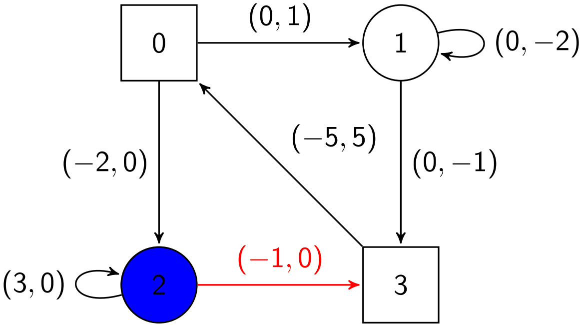
$c_1 = 5, c_2 = 2.$

What is a counter reachability game?



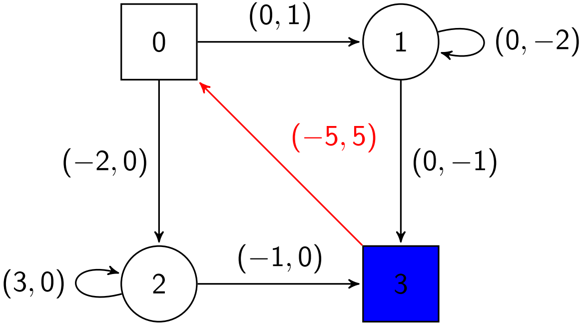
$c_1 = 3, c_2 = 2.$

What is a counter reachability game?



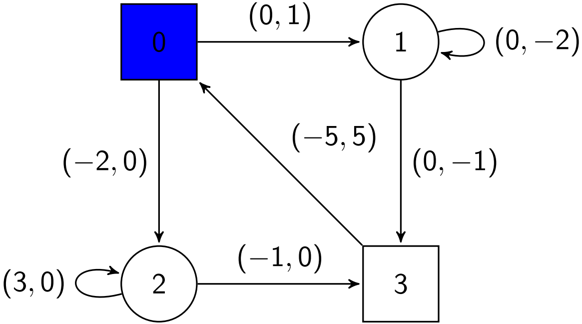
$c_1 = 6, c_2 = 2.$

What is a counter reachability game?



$c_1 = 5, c_2 = 2.$

What is a counter reachability game?



$c_1 = 0, c_2 = 7.$

Games on counter systems

- 2 players: Eve and Adam.
- States of a CS (with any semantics) partitioned into Eve's ($\bigcirc$) and Adam's ($\square$) states, forming the arena G.
- **Reachability game** played on G: (G, c_f) where $c_f \in Q \times \mathbb{Z}^d$ is the winning condition, (configuration that must appear in a run for Eve to win).
- **Reachability problems**: given (G, c_f) and c_0, decide if Eve has a strategy to reach c_f from c_0 in G.

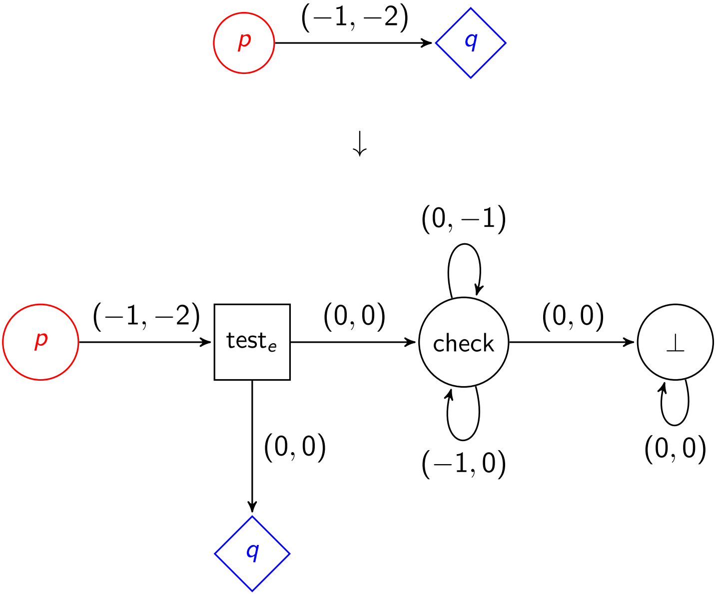
Outline

Counter reachability games in dimension two or more

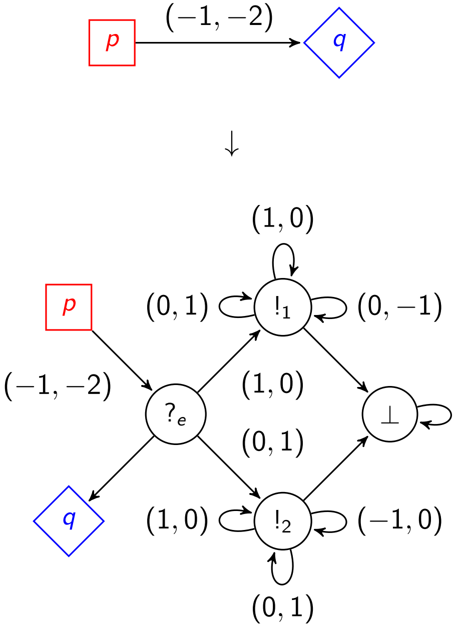
Counter reachability games in dimension one

- Relative integers semantics
- Non-blocking VASS semantics
- The case of zero-reachability on non-blocking VASS

Reduction from VASS to CS: first gadget.



Reduction from VASS to CS: second gadget.



Undecidability for VASS

In [BJK10], reachability games on VASS have been proved undecidable when:

- There are at least two counters;
- Integers in transitions are only ± 1 or 0.
- Objective: at least one counter is 0 while visiting $Z \subseteq Q$;

Undecidability for CS

The objective in [BJK10] can be transformed to the kind of objectives that we consider.

Theorem

Deciding the winner in counter reachability games is undecidable in dimension two.

Outline

Counter reachability games in dimension two or more

Counter reachability games in dimension one

Relative integers semantics

Non-blocking VASS semantics

The case of zero-reachability on non-blocking VASS

Reduction from CS to VASS

Required: integers in transitions are only ± 1 or 0.

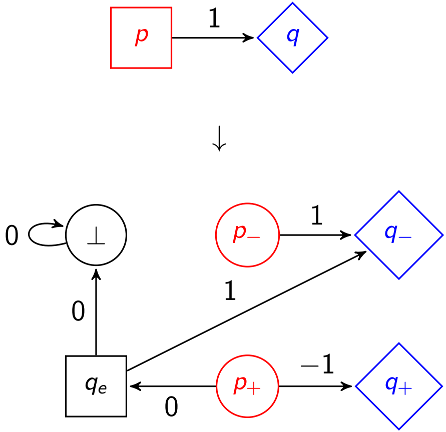
Idea: we simulate on a VASS the value $x \in \mathbb{Z}$ in a counter system with $|x|$ in two copies of the states, plus and minus.

From (Q, E) with objective $(q_f, 0)$, we build (Q', E'), where $Q' = Q_+ \cup Q_- \cup \{\perp\} \cup \{(\text{checking states})\}$, with objective $(\perp, 0)$.

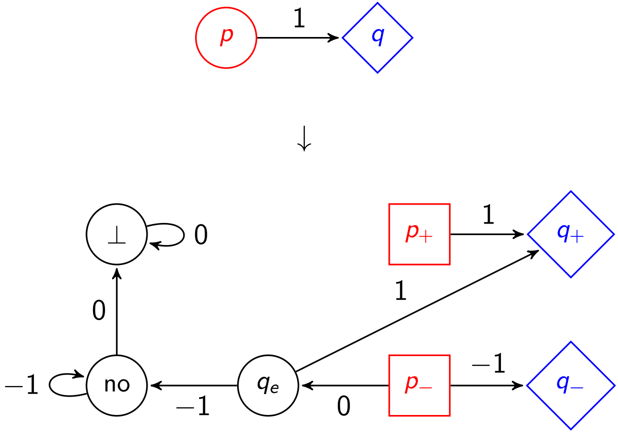
In E': gadgets to move from one copy to another.

Entering a gadget when counter value $\neq 0 \Rightarrow$ losing.

Reduction from CS to VASS: first gadget.



Reduction from CS to VASS: second gadget.



Complexity of CRG with the $\mathbb{Z}$ semantics

The reduction from VASS to CS still holds in dimension one.

In [BJK10], deciding the winner in the reachability games that we consider is PSPACE-complete in dimension one.

Theorem

Deciding the winner in counter reachability games with the $\mathbb{Z}$ semantics, zero-objective and short transitions is PSPACE-complete.

Outline

Counter reachability games in dimension two or more

Counter reachability games in dimension one

Relative integers semantics

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The case of zero-reachability on non-blocking VASS

Reduction from NBVASS to VASS

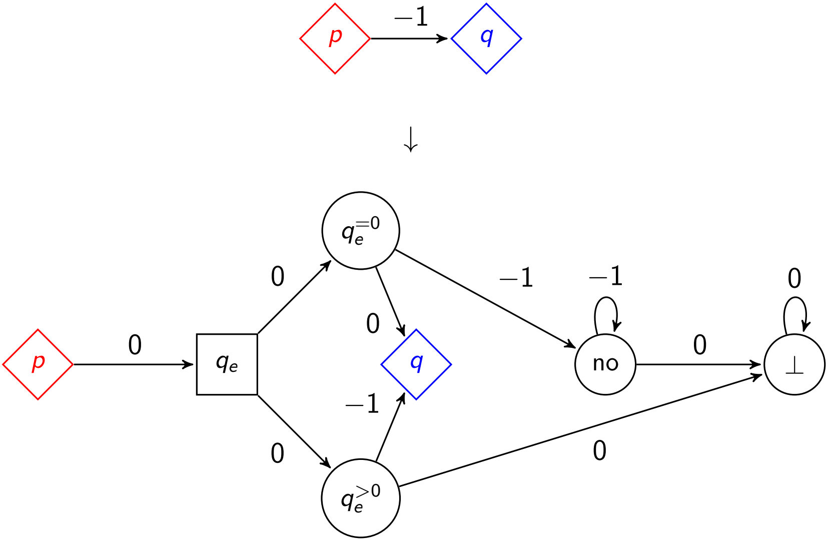
Required: integers in transitions are only ± 1 or 0.

Goal of the reduction: Simulate on a VASS "$0 - 1 = 0$".

Idea: Two moves for Adam instead of each -1 transition, one if counter value 0 and one else. Eve has a winning strategy in the two states iff Adam was wrong.

Objective: $(q_f, 1)$ in the NBVASS, $(\perp, 0)$ in the VASS.

Reduction from NBVASS to VASS: gadget



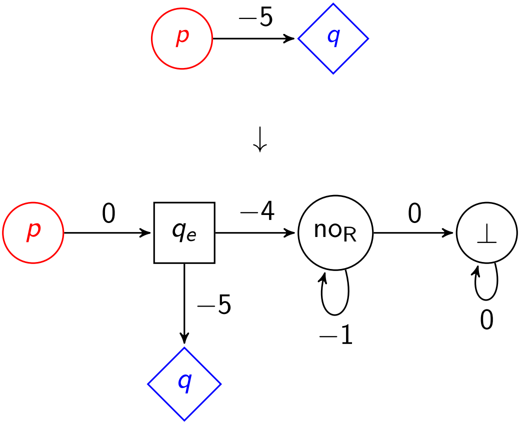
Reduction from VASS to NBVASS

Objective: $(q_f, 0)$ in the VASS, $(\perp, 1)$ in the NBVASS.

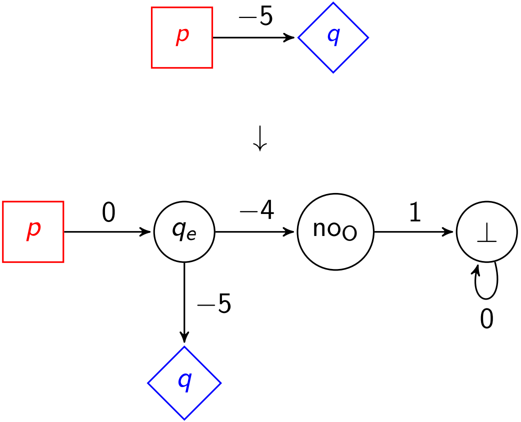
Idea: for each decreasing transition in the NBVASS, decreased amount $>$ counter value $\Rightarrow$ adversary has a winning strategy.

How: Going to $\perp$, gadgets ensure counter value 1 iff Eve was right.

Reduction from VASS to NBVASS: first gadget



Reduction from VASS to NBVASS: second gadget



Complexity of CRG with the NBVASS semantics

Theorem

Deciding the winner in counter reachability games with the NBVASS semantics, objective 1 *and short transitions is PSPACE-complete.*

If objective $x \in \mathbb{Z}$, binary representation $\Rightarrow$ exponential a priori.

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Reaching zero on a NBVASS with short transitions

- Eve's winning set is here downwards-closed.
- Hence, for each state, there is a maximal value s.t. Eve has a winning strategy (or Eve wins for no value).
- Consequence: If there is a winning value, then 0 is one of them $\Rightarrow$ set Q_Z of states, computable with a PTIME algorithm.
- We just have to decide in polynomial time ([BJK10]) whether 0 is reachable in the system restricted to Q_Z.

Theorem

Deciding the winner in counter reachability games with the NBVASS semantics, objective 0 *and short transitions is in P.*

Conclusion

- Three semantics for counter reachability games, roughly same complexity in dimension one.
- In dimension two, everything undecidable.
- Interesting gap for the NBVASS semantics depending on the objective.
- With arbitrary integers on transitions, complexity gap for decision problem: EXPTIME-hard and in EXPSPACE.

Thank you for your attention!