On The Complexity of Counter Reachability Games

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RP, September 27th, 2013
What is a counter system?

- **d-dimensional counter system (CS):** \((Q, E)\).

- Example: \(d = 2\), \(Q = \{0, 1, 2\}\), \(E = \{(0, (1, 1), 0), (0, (-2, 0), 1), (1, (0, -3), 2), (2, (0, 0), 2)\}\).

- **Configuration:** \(c_i = (q_i, (v^i_1, \ldots, v^i_d)) \in Q \times \mathbb{Z}^d\).

- **Run example:**
  
  \((0, (1, 0)) \rightarrow (0, (2, 1)) \rightarrow (1, (0, 1)) \rightarrow (2, (0, -2)) \rightarrow \ldots\)

  finite or not.
What is a counter system?

- $\mathbb{Z}$ semantics: counters are relative integers, no restriction on transitions.

- Vector Addition System with States (VASS): CS with an additional property: in each run, a transition is disabled if it would lead to a configuration with a negative counter value.

- Non-blocking VASS semantics (NBVASS): unlike for VASS, every transition is always enabled, negative counter values are immediately replaced by zero.
What is a counter reachability game?

\[ c_1 = 5, \quad c_2 = 2.\]
What is a counter reachability game?

\[ c_1 = 5, \quad c_2 = 2. \]
What is a counter reachability game?

\[
c_1 = 3, \ c_2 = 2.
\]
What is a counter reachability game?

c_1 = 6, c_2 = 2.
What is a counter reachability game?

$c_1 = 5, c_2 = 2.$
What is a counter reachability game?

\[ c_1 = 0, \ c_2 = 7. \]
2 players: Eve and Adam.

States of a CS (with any semantics) partitioned into Eve’s (◯) and Adam’s (□) states, forming the arena $G$.

Reachability game played on $G$: $(G, c_f)$ where $c_f \in Q \times \mathbb{Z}^d$ is the winning condition, (configuration that must appear in a run for Eve to win).

Reachability problems: given $(G, c_f)$ and $c_0$, decide if Eve has a strategy to reach $c_f$ from $c_0$ in $G$. 

Games on counter systems
Outline

Counter reachability games in dimension two or more

Counter reachability games in dimension one
  Relative integers semantics
  Non-blocking VASS semantics
  The case of zero-reachability on non-blocking VASS
Reduction from VASS to CS: first gadget.
Reduction from VASS to CS: second gadget.

\[ p \xrightarrow{(-1, -2)} q \]

\[ \downarrow \]

\[
\begin{array}{c}
p \\
\Diamond \\
\bullet \xrightarrow{(0, 1)} \blacksquare \\
\bullet \xrightarrow{(1, 0)} \bullet \xrightarrow{(0, -1)} \bullet \\
\bullet \xrightarrow{(0, 1)} \bullet \xrightarrow{(-1, 0)} \bullet \\
\bullet \xrightarrow{(1, 0)} \bullet \xrightarrow{(0, 1)} \bullet \\
\bullet \xrightarrow{(1, 0)} \bullet \xrightarrow{(0, 1)} \bullet
\end{array}
\]
Undecidability for VASS

In [BJK10], reachability games on VASS have been proved undecidable when:

- There are at least two counters;
- Integers in transitions are only $\pm 1$ or $0$.
- Objective: at least one counter is $0$ while visiting $Z \subseteq Q$;
The objective in [BJK10] can be transformed to the kind of objectives that we consider.

**Theorem**

*Deciding the winner in counter reachability games is undecidable in dimension two.*
Outline

Counter reachability games in dimension two or more

Counter reachability games in dimension one

Relative integers semantics

Non-blocking VASS semantics

The case of zero-reachability on non-blocking VASS
Reduction from CS to VASS

Required: integers in transitions are only ±1 or 0.

Idea: we simulate on a VASS the value \( x \in \mathbb{Z} \) in a counter system with \(|x|\) in two copies of the states, plus and minus.

From \((Q, E)\) with objective \((q_f, 0)\), we build \((Q', E')\), where \(Q' = Q_+ \cup Q_- \cup \{\bot\} \cup \{(\text{checking states})\}\), with objective \((\bot, 0)\).

In \(E'\): gadgets to move from one copy to another.

Entering a gadget when counter value \(\neq 0\) \(\Rightarrow\) losing.
Reduction from CS to VASS: first gadget.
Reduction from CS to VASS: second gadget.

\[
\begin{array}{c}
\text{\textcolor{red}{\(p\)}} \quad 1 \quad \text{\textcolor{blue}{\(q\)}} \\
\downarrow \\
\text{\textcolor{red}{\(\bot\)}} \quad 0 \quad \text{\textcolor{red}{\(p_{+}\)}} \quad 1 \quad \text{\textcolor{blue}{\(q_{+}\)}} \\
\text{\textcolor{red}{\(\text{\(no\)}} \quad 0 \quad \text{\textcolor{red}{\(p_{-}\)}} \quad -1 \quad \text{\textcolor{blue}{\(q_{-}\)}} \\
\end{array}
\]
Complexity of CRG with the $\mathbb{Z}$ semantics

The reduction from VASS to CS still holds in dimension one.

In [BJK10], deciding the winner in the reachability games that we consider is PSPACE-complete in dimension one.

**Theorem**

*Deciding the winner in counter reachability games with the $\mathbb{Z}$ semantics, zero-objective and short transitions is PSPACE-complete.*
Outline

Counter reachability games in dimension two or more

Counter reachability games in dimension one

Relative integers semantics

Non-blocking VASS semantics

The case of zero-reachability on non-blocking VASS
Reduction from NBVASS to VASS

Required: integers in transitions are only $\pm 1$ or 0.

Goal of the reduction: Simulate on a VASS “$0 - 1 = 0$”.

Idea: Two moves for Adam instead of each $-1$ transition, one if counter value 0 and one else. Eve has a winning strategy in the two states iff Adam was wrong.

Objective: $(q_f, 1)$ in the NBVASS, $(\bot, 0)$ in the VASS.
Reduction from NBVASS to VASS: gadget

\[
p \xrightarrow{-1} q
\]

\[
p \xrightarrow{0} q_e \xrightarrow{0} q_e = 0 \xrightarrow{-1} q_e > 0 \xrightarrow{-1} q_e = 0 \xrightarrow{0} \perp
\]

\[
q \xrightarrow{-1} no \xrightarrow{0} \perp
\]
Reduction from VASS to NBVASS

Objective: \((q_f, 0)\) in the VASS, \((\bot, 1)\) in the NBVASS.

Idea: for each decreasing transition in the NBVASS, decreased amount \(>\) counter value \(\Rightarrow\) adversary has a winning strategy.

How: Going to \(\bot\), gadgets ensure counter value 1 iff Eve was right.
Reduction from VASS to NBVASS: first gadget

\[
\begin{align*}
 & p \xrightarrow{-5} q \\
 & \downarrow \\
 & p \xrightarrow{0} q_e \xrightarrow{-4} noR \xrightarrow{0} \bot \\
 & \downarrow \quad \downarrow \quad \quad \downarrow \quad \downarrow \\
 & q \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \bot \\
\end{align*}
\]
Reduction from VASS to NBVASS: second gadget
Complexity of CRG with the NBVASS semantics

Theorem

*Deciding the winner in counter reachability games with the NBVASS semantics, objective 1 and short transitions is PSPACE-complete.*

If objective $x \in \mathbb{Z}$, binary representation $\Rightarrow$ exponential a priori.
Outline

Counter reachability games in dimension two or more

Counter reachability games in dimension one

Relative integers semantics

Non-blocking VASS semantics

The case of zero-reachability on non-blocking VASS
Reaching zero on a NBVASS with short transitions

- Eve’s winning set is here downwards-closed.
- Hence, for each state, there is a maximal value s.t. Eve has a winning strategy (or Eve wins for no value).
- Consequence: If there is a winning value, then 0 is one of them \( Q_Z \) of states, computable with a PTIME algorithm.
- We just have to decide in polynomial time ([BJK10]) whether 0 is reachable in the system restricted to \( Q_Z \).

**Theorem**

*Deciding the winner in counter reachability games with the NBVASS semantics, objective 0 and short transitions is in P.*
Conclusion

- Three semantics for counter reachability games, roughly same complexity in dimension one.

- In dimension two, everything undecidable.

- Interesting gap for the NBVASS semantics depending on the objective.

- With arbitrary integers on transitions, complexity gap for decision problem: EXPTIME-hard and in EXPSPACE.
Thank you for your attention!