Chapter 15: Dynamic Programming

Main idea, solving a problem

- Solve subproblems first
- Use solutions to solve main problem

Adding a basic idea: Never solve the same problem twice!

- Solve subproblems first
- Use solutions to solve main problem
- Each time we have an optimal solution to a subproblem, save it in a table
- When solving a (sub-) problem, first look in the table to see if it was already solved

Two initial examples

- Fibonacci numbers
  \[ F_0 = 0 \quad F_1 = 1 \quad F_i = F_{i-1} + F_{i-2} \]
  \[ 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots \]
- Assembly-line scheduling

Matrix Chain Multiplication

- \( A \) is a \( p \times q \) matrix
- \( B \) is a \( q \times r \) matrix
- \( AB \) is a \( p \times r \) matrix
- Dominating multiplication cost: \( pqr \) multiplications

Example

\[ A_1, A_2, A_4, A_4, A_3 \]
\[ A_1 = 30 \times 35 \]
\[ A_2 = 35 \times 15 \]
\[ A_4 = 15 \times 5 \]
\[ A_5 = 5 \times 10 \]
\[ A_4 = 10 \times 20 \]
\[ A_5 = 20 \times 25 \]
What is the total cost of the chain?

\(((A_1A_2A_3)(A_4(A_5A_6)))\)

<table>
<thead>
<tr>
<th>(A_i)</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>(30\times35)</td>
<td>1050</td>
</tr>
<tr>
<td>(35\times15)</td>
<td>525</td>
</tr>
<tr>
<td>(15\times5)</td>
<td>75</td>
</tr>
<tr>
<td>(5\times10)</td>
<td>50</td>
</tr>
<tr>
<td>(10\times20)</td>
<td>200</td>
</tr>
<tr>
<td>(20\times25)</td>
<td>500</td>
</tr>
<tr>
<td>TOTAL</td>
<td>4050</td>
</tr>
</tbody>
</table>

How about if we multiply in the reverse order?

\((A_1(A_4(A_5A_6)))\)

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<td>200</td>
</tr>
<tr>
<td>(20\times25)</td>
<td>500</td>
</tr>
<tr>
<td>TOTAL</td>
<td>4750</td>
</tr>
</tbody>
</table>

...or, one of these orders?

\(((A_1A_2A_3)(A_4(A_5A_6)))\)?

\((A_1A_2)(A_3A_4)(A_5A_6))\)?

Finding the optimal parenthesization

- Divide the problem in two parts
- Solve each part optimally
- The only problem is that the problem can be divided in more than one way

Finding the optimal parenthesization

- We need to know the optimal solutions to the subproblems in order to choose

Finding the optimal parenthesization

- Solve all possible subproblems first
- Each time we have an optimal solution to a subproblem, save it in a table
- When solving a (sub-) problem, first look in the table to see if it already solved
The Dynamic Programming solution

<table>
<thead>
<tr>
<th>A_1</th>
<th>A_2</th>
<th>A_3</th>
<th>A_4</th>
</tr>
</thead>
<tbody>
<tr>
<td>30x35</td>
<td>15750</td>
<td>30x5</td>
<td>30x10</td>
</tr>
<tr>
<td>35x15</td>
<td>2625</td>
<td>35x5</td>
<td>35x10</td>
</tr>
<tr>
<td>15x5</td>
<td>750</td>
<td>15x10</td>
<td>15x25</td>
</tr>
<tr>
<td>5x10</td>
<td>1000</td>
<td>5x20</td>
<td>5x25</td>
</tr>
<tr>
<td>20x25</td>
<td>5000</td>
<td>10x25</td>
<td></td>
</tr>
</tbody>
</table>

What is the cost of multiplying (A_1 A_2 A_3)?
1: (A_1 A_2) A_3
2: A_1 (A_2 A_3)
3: A_1 A_2 A_3

1: 15750 + cost of multiplying (A_1 A_2) 30x15 with A_3 15x5 = 15750 + 30x15x5 = 18000
2: 2625 + cost of multiplying A_2 30x35 with (A_1 A_3) 35x5 = 2625 + 30x35x5 = 7875

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