Chapter 28: Matrix Multiplication

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$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$x = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix} \qquad x^T = \begin{pmatrix} 4 & 3 & 6 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Diagonal matrix $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

Identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

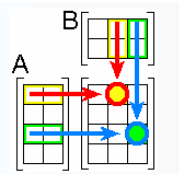
Permutation matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Upper - triangular matrix $\begin{pmatrix} 4 & 5 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 6 \end{pmatrix}$

Multiplying Matrices C=A*B

Each cell in C is computed from one row in A and one column in B

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

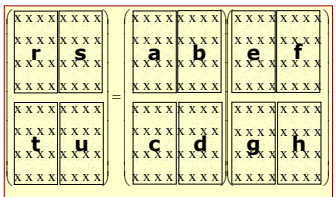


Multiplying 2 2x2 matrices in 8 multiplications

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$r = ae + bg$$
$$s = af + bh$$
$$t = ce + dg$$
$$u = cf + dh$$

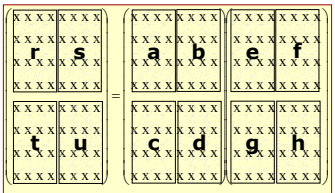
This method can be applied recursively



$r = ae + bg$
$s = af + bh$
$t = ce + dg$
$u = cf + dh$

Informationsteknologi

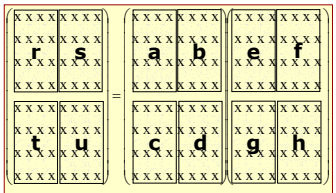
Multiply 2 n x n matrices by 8 multiplications of n/2 x n/2 matrices, adding the results



$$r = ae + bg$$
$$s = af + bh$$
$$t = ce + dg$$
$$u = cf + dh$$

Informationsteknologi

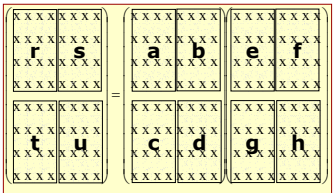
T(n) = 8 T(n/2) + Θ(n²)



$$r = ae + bg$$
$$s = af + bh$$
$$t = ce + dg$$
$$u = cf + dh$$

Informationsteknologi

Strassen's observation: We only need 7 multiplications!
T(n) = 7 T(n/2) + Θ(n²)



$$r = ae + bg$$
$$s = af + bh$$
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Informationsteknologi

7 multiplications:

$$P_1 = a \cdot (f - h) = af - ah$$
$$P_2 = (a + b) \cdot h = ah + bh$$
$$P_3 = (c + d) \cdot e = ce + de$$
$$P_4 = d \cdot (g - e) = dg - de$$
$$P_5 = (a + d) \cdot (e + h) = ae + ah + de + dh$$
$$P_6 = (b - d) \cdot (g + h) = bg + bh - dg - dh$$
$$P_7 = (a - c) \cdot (e + f) = ae + af - ce - cf$$

$$r = P_5 + P_4 - P_2 + P_6 = ae + bg$$
$$s = P_1 + P_2 = af + bh$$
$$t = P_3 + P_4 = ae + de$$
$$u = P_5 + P_1 - P_3 - P_7 = cf + dh$$