Chapter 4: Recurrences

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Today's lectures

Textbook chapters 1-3

- Introduction to Recurrences
- The Substitution Method
- The Recursion Tree Method
- The Master Method (Master Theorem)
- Exercises & Problems

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Textbook chapter 4

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Why do we need recurrences?

- A recursive equation corresponds to the standard algorithmic technique of divide-and-conquer:
  1. Divide the problem into subproblems
  2. Solver (some of) the subproblems recursively
  3. Combine the solutions to the subproblems

$$T(\text{problem}) = T_{\text{Divide and Combine}}(\text{problem}) + \sum_{\text{Subproblems}} T(\text{subproblem})$$

Examples

- Binary search

$$\begin{cases} T(n) = T(n/2) + \Theta(1) \\ T(1) = \Theta(1) \end{cases}$$

$$T(n) = T(n/2) + \Theta(1)$$

- Merge Sort

$$T(n) = 2T(n/2) + \Theta(n)$$

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General Method

1. Guess the form of the solution
2. Use induction to
   - Find constants in solution
   - Prove the solutioon

Some tricks

1. Try to ignore all ”unimportant” constants and lower-order terms
2. If things only ”almost” work out, try to extend the guess by subtracting a constant term
3. Change (substitute) variables

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General Method

1. Develop the recursive equation a number of levels
2. If possible, solve by concluding how the entire sum will look like
3. Otherwise, inspect a number of levels, see the tendency, and guess a solution, which can then be proven by other methods

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The Master Theorem

Master Theorem

$T(n) = aT(n/b) + f(n)$ has the following solutions

1. $f(n) = O(n^{\log_b a-\varepsilon}) \quad \Rightarrow \quad T(n) = \Theta(n^{\log_b a})$
2. $f(n) = \Theta(n^{\log_b a}) \quad \Rightarrow \quad T(n) = \Theta(n^{\log_b a} \log n)$
3. $\begin{cases} f(n) = \Omega(n^{\log_b a+\varepsilon}) \\ af(n/b) \le cf(n) \end{cases} \quad \Rightarrow \quad T(n) = \Theta(f(n))$

Looks complicated, but let's try to understand it

Master Theorem

$T(n) = \underbrace{aT(n/b)}_{A} + \underbrace{f(n)}_{B}$ has the following solutions

1. $f(n) = O(n^{\log_b a - \varepsilon})$ A dominates $\Rightarrow T(n) = \Theta(n^{\log_b a})$
2. $f(n) = \Theta(n^{\log_b a})$ A and B balances $\Rightarrow T(n) = \Theta(n^{\log_b a} \log n)$
3. $\begin{cases} f(n) = \Omega(n^{\log_b a + \varepsilon}) \\ af(n/b) \le cf(n) \end{cases}$ B dominates $\Rightarrow T(n) = \Theta(f(n))$

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