Analysis of numerical methods  
Assignment 1

1. Discretise the advection equation with periodic boundary conditions,
\[ u_t + u_x = 0, \quad 0 < x < 2\pi, t > 0, \]
\[ u(0, t) = u(2\pi, t), \quad t \geq 0, \]
\[ u(x, 0) = f(x), \quad 0 \leq x \leq 2\pi, \]
in space using fourth order central finite differences. Do not discretise in time.

(a) Derive and plot the wave speed and the relative error in wave speed as a function of $\omega h$.

(b) Derive the leading term of the maximum point-wise error at time $t$ given a single-frequency initial condition, i.e. $f = e^{i\omega x}$.

2. Consider the system of partial differential equations,
\[ u_t = Au_x + Bu_{xx}, \quad 0 < x < 2\pi, t > 0, \]
\[ u(0, t) = u(2\pi, t), \quad t \geq 0, \]
\[ u(x, 0) = f(x), \quad 0 \leq x \leq 2\pi, \]
where
\[ u = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \frac{1}{2} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \]

Discretise it in time and space with the Crank-Nicolson method and derive a stability condition for the time step $k$. 