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Examination in Analysis of Numerical Methods, October 19 2011

Time: $08^{00} - 13^{00}$
Tools: Mathematics Handbook ("Beta"), calculator
Maximum number of points is 30. To pass you need at least 13 points. To get all points on a question you must support your answers by detailed computations and motivated assumptions. Introduced notation must be defined.

1. (6p) Consider the problem

$$u_t + Au_x = 0, \quad A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{1}$$

on the domain $x \in [\,0\,,1\,]$, $t \geq 0$ with initial data $u(x,0) = u_0(x)$.

a) To begin with consider the periodic problem, and the explicit method

$$u_j^{n+1} = (I - kAD)u_j^n.$$

Here k is the time step and u_j^n is a grid function on an equidistant grid with spatial step h, approximating the solution at $t = nk$. By using different operators D we get different methods. Determine which (if any) of the following D's can be used, and if relevant, how the time step should be choosen.

$$D_1 = D_0, \quad D_2 = D_+, \quad D_3 = D_-.$$

b) Which of the following boundary conditions yield a well-posed IBVP?

$$(i) \begin{cases} u_1(0,t) = 0 \\ u_2(1,t) = 0 \end{cases} \quad (ii) \begin{cases} u_1(0,t) = 0 \\ u_2(0,t) = 0 \end{cases} \quad (iii) \begin{cases} u_1(0,t) = -u_2(0,t) \\ u_1(1,t) = -u_2(1,t) \end{cases}$$

2. (6p) Consider the Fourier method for solving the 2π-periodic problem

$$u_t = (b(x)u)_x, \tag{2}$$

with initial data $u(x,0) = f(x)$, on an equidistant grid with step $h = \frac{2\pi}{N}$.

a) Given a periodic grid function u_j, $j = 0, \dots N-1$, describe how the right hand side of (2) is efficiently approximated in the Fourier method. You don't need to include details of the FFT algorithm here.

b) The FFT algorithm is based on efficient evaluation of

$$a(y) = a_0 + a_1 y + a_2 y^2 + \cdots + a_{N-1} y^{N-1},$$

for the N different $y_j = e^{-ij\theta}$, $j = 0, \dots N-1$. Here $\theta = 2\pi/N$.

Assume $N = 2M$. Show how the evaluation of the $a(y)$ for the N different $e^{-ij\theta}$ can be done by evaluating 2 polynomials of degree M for M arguments each. In a second step these $M + M$ values are combined to obtain the N different a-values. How is this done efficiently?

3. (6p) Consider

$$u_t + (f(u))_x = 0, \quad f(u) = u^4.$$

a) Show that the following Lax-Wendroff type discretization is conservative,

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f_{j+1}^n - f_{j-1}^n\right) + \frac{\lambda^2}{2}\left(D_{j+1}^n(f_{j+1}^n - f_j^n) - D_j^n(f_j^n - f_{j-1}^n)\right).$$

Here $f_j^n = f(u_j^n)$, $D_j^n = f'(u_j^n)$ and $\lambda = k/h$.

b) Determine the entropy satisfying solution of the Riemann problem with initial data

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0, \\ \text{-1} & \text{if } x > 0. \end{cases}$$

c) Linearize around $u = 1$ and derive an expression for the dispersion error for the above method when $\lambda = 0.2$. What quantity determines if waves travel "too fast" or "too slow"?

4. (6p) Consider

$$u_t = u_{xx},\ u(x,0) = f(x), \quad 0 \le x \le 1, \quad u(0,t) = 0, \quad u(1,t) = 0. \qquad (3)$$

a) Derive an energy estimate for the solution.

b) Let v_j, w_j be grid functions on an equidistant grid with spatial step $h = 1/(N+1)$. Show that

$$\Sigma_1^N h v_j D_+ w_j = -v_0 w_1 - \Sigma_1^N h(D_- v_j) w_j + v_N w_{N+1}. \qquad (4)$$

c) Consider a semi-discretization of (3) in space using D_+D_-. Assume the boundary conditions are imposed directly, that is

$$u_0(t) = u_{N+1}(t) = 0.$$

Show an energy estimate for the semi-discrete problem. You may use (4).

d) The 6^{th}−order backward differencing method is suggested for the time discretization. The stability domain of this method includes the negative real axis. Is the resulting fully discrete problem stable for all choices of time step k? Motivate your answer.

5. (6p) Consider the IBVP on $0 \leq x \leq 1$ for

$$u_t + u_x = F, \quad u(0,t) = g(t), \quad u(x,0) = f(x). \tag{5}$$

Introduce grid functions $\mathbf{u}(t) = \{u_i(t)\}, u_i(t) \approx u(ih,t), i = 0, 1.., N, h = 1/N$, and D_1, defined by,

$$D_1 v_j = \begin{cases} \mathrm{D}_+ v_j \text{ if } j = 0 \\ \mathrm{D}_0 v_j \text{ if } 0 < j < N \\ \mathrm{D}_- v_j \text{ if } j = N \end{cases}$$

is an SBP operator with respect to the inner product

$$\langle \mathbf{v}, \mathbf{w} \rangle_{H,h} = \frac{h}{2} v_0 w_0 + \Sigma_1^{N-1} h v_i w_i + \frac{h}{2} v_N w_N.$$

The corresponding norm is $\| \cdot \|_{H,h}^2 = \langle \cdot, \cdot \rangle_{H,h}$.

a) Consider the semi-discretization

$$\frac{d\mathbf{u}}{dt} = -D_1 \mathbf{u} - \frac{\sigma}{h} H^{-1} \begin{pmatrix} u_0 - g \\ 0 \\ \vdots \end{pmatrix} + \mathbf{F}(t).$$

Here $u_i(0) = f(x_i)$ and $\mathbf{F}(t)$ is the grid function corresponding to $F(x,t)$. Determine the penalty parameter σ so that an energy estimate for the case $g = F = 0$ is valid.

b) Determine the truncation error and derive a semi-discrete equation for the point-wise error $e_i(t) = u(x_i, t) - u_i(t)$.

c) Given an energy estimate for the semi-discrete problem with $F \neq 0$, $g = 0$, $f \neq 0$ of the form

$$\|\mathbf{u}(t)\|_{H,h} \leq \|\mathbf{f}\|_{H,h} + \int_0^t \|\mathbf{F}(s)\|_{H,h} ds, \tag{6}$$

give an energy estimate of the error, i.e. prove convergence of the semi-discrete solution to the continuous solution. At what rate does the norm of the error go to zero?

Good Luck !