1) \[ \ddot{u}_t + Au_x = \ddot{0}, \quad 0 < x < 1, \quad t > 0 \]

\[ A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \]

a) Determine the eigenvalues of \( A \).

\[ \det(A - \lambda I) = 0 \]

\[ 0 = \begin{vmatrix} 2 - \lambda & 3 \\ 3 & 2 - \lambda \end{vmatrix} = (\lambda - 2)^2 - 9 \]

\[ \lambda = 2 \pm 3 \quad \{ \begin{array}{c} \lambda_1 = 5 \\ \lambda_2 = -1 \end{array} \} \]

Since the eigenvalues are real and distinct, there exists an orthogonal transform which diagonalizes \( A \).

Let \( \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = V^{-1}AV, \; V^{-1} = V^T \)

and define the characteristic variables

\[ \vec{W} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = V^{-1} \vec{u}. \]

Note that the difference operators \( D \) commute with non-singular matrices.
We have the difference scheme
\[ \tilde{U}^{n+1}_{j} = (I - kAD)\tilde{U}^{n}_{j}. \]
Multiply from the left by \( V_{j}^{-1} \), and insert the identity matrix at suitable places,
\[ V_{j}^{-1}\tilde{U}^{n+1}_{j} = V_{j}^{-1}(VV^{-1} - kAD)VV^{-1}\tilde{U}^{n}_{j}. \]
\[ \tilde{W}^{n+1}_{j} = (I - k V^{-1} A D V)\tilde{W}^{n}_{j}. \]
\[ \tilde{W}^{n+1}_{j} = (I - k A D)\tilde{W}^{n}_{j}. \]
We have diagonalised the system and decoupled the equations. We thus have two independent equations
\[ (\tilde{W}_{i})^{n+1}_{j} = (\tilde{W}_{i})^{n}_{j} - k \lambda_{i} D(\tilde{W}_{i})^{n}_{j}. \]
The CFL condition, that the numerical domain of dependence must include the analytical domain of dependence, is necessary for stability. We draw the stencils for $D_+$ and $D_-$ in a characteristics plot.

We see that for $D_+$, the domain of dependence for $w_1$ will fall out of the numerical domain of dependence regardless of the time step. For $D_-$, the domain of dependence for $w_2$ will fall out of the numerical domain of dependence regardless of the time step.

Both $D_+$ and $D_-$ thus yield unstable discretisations.
For \( D_0 \) it is possible to fulfill the CFL-condition. Since the condition is not sufficient, we need another method of analysis.

Express \( w_i \) through the inverse discrete Fourier transform,

\[
(w_i)_j^N = \sum_{\omega} (\hat{w}_\omega)_j^N e^{i\omega x_j}
\]

Since the equation is linear we can treat each frequency independently.

Make the ansatz

\[
(w_i)_j^N = \hat{w}_\omega e^{i\omega x_j}
\]

and insert it into the scheme with \( D = D_0 \).

\[
\hat{w}_{\omega}^{n+1} e^{i\omega x_j} = \hat{w}_\omega^n e^{i\omega x_j} - \frac{k}{\Delta h} \lambda_i \hat{w}_\omega^n (e^{i\omega (x_{j+1})} - e^{i\omega (x_j)})
\]

\[
\hat{w}_{\omega}^{n+1} = \hat{w}_\omega^n - \frac{k}{\Delta h} \lambda_i \hat{w}_\omega^n (e^{i\omega h} - e^{-i\omega h})
\]

\[
\hat{w}_{\omega}^{n+1} = \hat{w}_\omega^n - \frac{ik}{\Delta h} \lambda_i \sin(\omega h) \hat{w}_\omega^n
\]

Stability if \( |\hat{w}_{\omega}^{n+1}| \leq |\hat{w}_\omega^n| \quad (5)\)

\[
1 - \frac{ik}{\Delta h} \lambda_i \sin(\omega h) \leq 1 \quad , \quad i = 1, 2
\]

This is violated for all frequencies \( \omega 
eq 0 \).

All three proposed schemes are thus unstable.
b) The characteristics look like:

We need B.C. for
\( W_1 \) at \( x = 0 \) and for
\( W_2 \) at \( x = 1 \)

\[
\begin{align*}
W_1 &= \frac{1}{\sqrt{2}} (U_1 + U_2) \\
W_2 &= \frac{1}{\sqrt{2}} (U_1 - U_2)
\end{align*}
\]

\( U_1 \) is well-posed.

(i) \( U_1(0,t) = 0 \)
\( W_1(0,t) = -W_2(0,t) - \alpha k \)
\( U_2(1,t) = 0 \)
\( W_2(1,t) = W_1(1,t) + \alpha k \)

(ii) Gives no boundary data at \( x = 1 \) \( \Rightarrow \) ill-posed.

(iii) \( U_1(0,t) + U_2(0,t) = 0 \)
\( W_1(0,t) = 0 \) \( - \alpha k \)
\( U_1(1,t) + U_2(1,t) = 0 \)
\( W_1(1,t) = 0 \) \( \text{need for } W_2 \)

\( \Rightarrow \) ill-posed.
2) \( u_t = (a(x) u)_x, \quad 0 \leq x \leq 2\pi, \quad t > 0 \)

periodic boundary conditions,

\( h = \frac{2\pi}{N}, \quad x_j = jh, \quad j = 0, 1, \ldots, N-1, \)

\( u_j \sim u(x_j). \)

a) Let \( v_j = a(x_j)u_j \),

\[
\hat{V}_k = \frac{1}{N} \sum_{j=0}^{N-1} v_j e^{-2\pi i jk/N} \quad \text{(DFT)}
\]

\[
v_j = \frac{1}{N} \sum_{k=-N/2+1}^{N/2} \hat{V}_k e^{2\pi i jk/N} \quad \text{(IDFT)}
\]

To evaluate \( (a(x)u)_x \) approximately,

1) Calculate \( v_j = a(x_j)u_j \).

2) Apply DFT to get \( \hat{V}_k \).

3) Differentiate in Fourier space,

\[
\frac{\partial}{\partial x_j} v_j = \frac{\partial}{\partial x_j} \frac{1}{N} \sum_{k=-N/2+1}^{N/2} \hat{V}_k e^{ikx_j} = \frac{1}{N} \sum_{k=-N/2+1}^{N/2} (i k \hat{V}_k) e^{ikx_j}
\]

4) Apply IDFT to \( ik \hat{V}_k \).
b) The discrete Fourier transform reads

\[ a(y_j) = \sum_{k=0}^{N-1} a_k \ y_j^k, \quad j = 0, \ldots, N-1, \quad N = 2M \]

\( a_k \) are (typically) point-wise values of a function, and

\[ y_j^k = e^{-2\pi i j k / N} \]

are the Fourier basis functions evaluated at point \( j \). The sum is split as

\[ a(y_j) = \sum_{k=0}^{M-1} \left( a_k \ y_j^k \right) + y_j \sum_{k=0}^{M-1} a_{k+M} \ y_j^k, \quad j = 0, \ldots, M-1. \]

With this expression we can evaluate

\[ a(y_j) = (1)_j + (2)_j, \quad j = 0, \ldots, M-1. \]

By noting that

\[ y_j^{M+j} = e^{-2\pi i j (M+N) / N} = y_j^{M} e^{-2\pi i j / N} = -y_j^j \]

we can construct the rest of the Fourier coefficients,

\[ a(y_j) = (1)_j - (2)_j, \quad j = M, \ldots, N-1. \]
3. a) **Definition:** A FD scheme for the conservation law

\[ u_t + f(u)_x = 0 \]

is conservative if it can be written as

\[ u_j^{n+1} = u_j^n - \frac{k}{h} \left( F_j^{n+\frac{1}{2}} - F_j^{n-\frac{1}{2}} \right), \]

where \( F \) is Lipschitz continuous and \( F(u, u, \ldots, u) = f(u) \).

This is fulfilled for the given scheme

\[ u_j^{n+1} = u_j^n - \frac{k}{2h} (f_j^n - f_{j-1}^n) + \frac{k^2}{2h} \left( D_{j+\frac{1}{2}}^n (f_j^n - f_{j-1}^n) + \right. \]

\[ \left. - D_{j-\frac{1}{2}}^n (f_j^n - f_{j-1}^n) \right), \]

by

\[ F_{j+\frac{1}{2}}^n = \frac{1}{2} (f_j^n + f_{j+1}^n) - \frac{k}{2h} (D_{j+\frac{1}{2}}^n - f_j^n). \]
b) \( u_t + (f(u))_x = 0 \) \quad f(u) = u^4

\[ U(x, t) = \begin{cases} U_L & x < 0 \\ U_R & x > 0 \end{cases} \]

Initially we have a shock at \( x = 0 \).

The shock propagation speed is given by the Riemann-Hugoniot condition,

\[
S = \frac{(U_L) - f(U_L))}{U_L - U_R} = \frac{u^4 - (-1)^4}{0} = 0.
\]

The entropy condition,

\[ 4 + 4 \cdot 1^3 = f'(U_L) > S > f'(U_R) = 4 \cdot (-1)^3 = -4, \]

is fulfilled. The solution will therefore have a shock also for \( t > 0 \).

Wherever \( u_x = 0 \), i.e. away from \( x = 0 \), \( u_x = 0 \).

The solution to the Riemann problem is then,

\[ U(x, t) = U(x, 0) = \begin{cases} 1 & x < 0 \\ -1 & x > 0 \end{cases} \]
c) The linearised equation is

\[ U_t + \lambda U_x = 0, \]  

It has the analytical solution

\[ u(x,t) = U(x - \lambda t, 0), \]

i.e., waves travel to the right with speed \( \lambda \) length units per time unit.

Denote the linearised flux function by \( \hat{f}(u) = \lambda u \). Let \( \hat{f}_j^n = \hat{f}(U_j^n) \), and \( \hat{D}_{ij} = \hat{f}_j^n \). Insert this into the scheme, \( U_j^{n+1} \)

\[ U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} \left( \hat{D}_{j+1}^n - \hat{D}_j^n \right) \]

\[ + \frac{\alpha_2}{2} \left( \hat{D}_{j+1}^n \left( \hat{D}_j^n - \hat{f}_j^n \right) - \hat{D}_j^n \left( \hat{D}_j^n - \hat{f}_{j-1}^n \right) \right) \]

\[ = U_j^n - 2\lambda (U_{j+1}^n - u_j^n) + \]

\[ + \frac{\alpha_2}{2} \left( U_{j+1}^n - 2U_j^n + U_{j-1}^n \right). \]

Make the ansatz

\[ U_j^n = \hat{u} \exp(\lambda x_j t). \]

The analytical solution to (*) would then be

\[ u(x_j, t) = \hat{u} \exp(\lambda (x_j - \lambda t)). \]
Inserting the ansatz yields

\[ u^{n+1} = u^n e^{i \omega x_j} - 2 \lambda u^n \left( e^{i \omega (x_j - h)} - e^{i \omega (x_j + h)} \right) + 8 \lambda^2 u^n \left( e^{i \omega (x_j - h)} - e^{i \omega x_j} + e^{i \omega (x_j + h)} \right) \]

\[ \hat{u}^{n+1} = \left( 1 - 2 \lambda (e^{i \omega h} - e^{-i \omega h}) + 8 \lambda^2 (e^{i \omega h/2} - e^{-i \omega h/2})^2 \right) \hat{u}^n \]

\[ \hat{u}^{n+1} = \left( 1 - 4i \lambda \sin(\omega h) - 32 \lambda^2 \sin^2 \left( \frac{\omega h}{2} \right) \right) \hat{u}^n \]

\[ g(\omega h) \]

\[ \hat{u}^{n+1} = \left| g \right| \exp \left( i \arg_{\hat{g}} \right) \hat{u}^n \]

\[ u^{n+1} = u^n e^{i \omega x_j} \]

\[ = \left| g \right| \hat{u}^n \exp \left( i \omega x_j + i \arg_{\hat{g}} \right) \]

\[ = \left| g \right| \hat{u}^n \exp \left( i \omega \left( x_j + \frac{\arg_{\hat{g}}}{\omega} \right) \right) \]

The correct solution at \( t_{n+1} \), given an exact \( u_j^{n+1} \), would be

\[ U(x_j, t_{n+1}) = \hat{u}^n e^{i \omega (x_j - qk)} \]

Thus, the deviation of \( \left| g \right| \) from 1 is the damping error, and the deviation of \( \frac{\arg_{\hat{g}}}{\omega} \) from -1 is the dispersion error.
\[
\begin{aligned}
\frac{\text{Arg} \, g}{\omega k} &= \frac{1}{\omega k} \text{ArcTan} \left( \frac{-4 \pi \sin(\omega h)}{1 - 32 \pi^2 \sin^2(\frac{\omega h}{2})} \right) \\
\text{Asymptotically, as } \omega h \to 0 \\
&\lim_{\omega h \to 0} \frac{\text{Arg} \, g}{\omega k} = \\
&= \lim_{\omega h \to 0} \frac{1}{\omega h} \text{ArcTan} \left( \frac{-4 \pi \sin(\omega h)}{1 - 32 \pi^2 \sin^2(\frac{\omega h}{2})} \right) \\
&= \lim_{\omega h \to 0} \frac{1}{\omega h} \text{ArcTan} \left( -4 \pi \sin(\omega h) \right) \\
&= \lim_{\omega h \to 0} \frac{1}{\omega h} \frac{-4 \pi \sin(\omega h)}{\omega h} = -4 \pi, \\
i.e., \text{ we get the correct wave speed.} \\
\text{As } \omega h \to \pi, \ g(\omega h) \text{ gets real, and assuming } 32 \pi^2 < 1, \text{ which is necessary for stability, } \text{Arg} \, g \to 0. \text{ Therefore, the highest frequency representable gets the wave speed } 0.\]

\[ 4. \begin{cases} u_t = u_{xx}, & 0 < x < 1, \ t > 0 \\
u(0,t) = u(1,t) = 0 \\
u(x,0) = f(x) \end{cases} \]

\[ a) \frac{d}{dt} \|u\|^2 = 2(u,u_t) = 2(u, u_{xx}) = 2 uu_x \bigg|_0^1 - 2(u_x, u_x) = -2 \|u_x\|^2 \leq 0. \quad \Box \]

\[ \Rightarrow \|u(\cdot, t)\| \leq \|f\|. \]

\[ b) \sum_{j=1}^{N} v_j D_+ w_j + \sum_{j=1}^{N} w_j D_- v_j = \]

\[ = \frac{1}{\Delta t} \sum_{j=1}^{N} \left( v_j v_{j+1} - v_j v_j + w_j v_j - w_j v_{j-1} \right) \]

\[ = \frac{1}{\Delta t} \left( \sum_{j=1}^{N} v_j w_{j+1} + v_N w_{N+1} - v_0 w - \sum_{j=1}^{N} v_j w_j \right) \]

\[ = -\frac{1}{\Delta t} v_0 w_1 + \frac{1}{\Delta t} v_N w_N. \quad \Box \]
c) Introduce the grid function $U_j(t)$ on the same mesh. $U_0(t) = U_{N+1}(t) = 0$ through injection. Spatial derivatives by $D_+D_-$. The semidiscretisation then reads,

\[
\begin{aligned}
\frac{d U_j}{dt} &= D_+D_-U_j + j = 1, \ldots, N \\
U_0 &= U_{N+1} = 0
\end{aligned}
\]

\[
\frac{d}{dt} \|U\|_2^2 = 2 \langle U, \dot{U} \rangle = 2 \langle \dot{U}, D_+D_-U \rangle,
\]

\[
\begin{aligned}
\text{Let } \mathbf{V}_j &= D_-U_j \} \\
&= 2 \sum_{j=1}^{N} U_j D_+D_-U_j = \frac{2}{h} U_0 V_1 - 2 \sum_{j=1}^{N} V_j D_-U_j + \frac{2}{h} U_N V_{N+1}
\end{aligned}
\]

\[
= -\frac{2}{h^2} \sum_{j=1}^{N} (u_j - u_{j-1})^2 + \frac{2}{h^2} u_N (u_{N+1}^2 - u_N^2)
\]

\[
= -\frac{2}{h^2} \sum_{j=1}^{N} (u_j - u_{j-1})^2 - \frac{2}{h^2} U_N^2 \leq 0.
\]
d) The semi-discretisation can be written
\[
\frac{d}{dt} \hat{U}(t) = A \hat{U}(t), \quad \text{where} \quad \hat{U} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.
\]

Since $A$ is symmetric its eigenvalues are real, and the stability estimate derived in c) shows that $A$ is negative definite. Thus, all its eigenvalues lie on the negative real axis, which is in the stability domain of the integrator. The method is thus stable.
5) \[ u_t + u_x = F(x,t), \quad 0 < x < 1, \quad t > 0 \]
\[ u(x,0) = f(x) \]
\[ u(0,t) = g(t) \]
\[ \hat{u}(t) = \left[ \hat{u}_i(t) \right]_i \quad \text{grid function at} \ x_i = i h, \]
\[ i = 0, \ldots, N. \]

\[ D_i \text{ operator s.t.} \]
\[ D_i V_j = \left\{ \begin{array}{ll}
D_x V_j & \text{if } j = 0, \\
D_x V_j & \text{if } j = 1, \ldots, N-1, \\
D_x V_j & \text{if } j = N.
\end{array} \right. \]

Corresponding inner product and norm given by the diagonal matrix \( H \):
\[ \langle \hat{u}, \hat{v} \rangle_{H^{-1}} = \sum_{i=1}^{N} \hat{u}_i \hat{v}_i = \frac{1}{2} u_0 v_0 + \sum_{i=1}^{N-1} u_i v_i + \frac{1}{2} u_N v_N. \]

a) Discretise the PDE in space by
\[ \frac{d}{dt} \hat{u} = -D_i \hat{u} - \frac{\sigma}{h} H (u_0 - g(t)) \hat{u}_0 + \hat{F}(t), \]
where \( \hat{u}_0 = \left( \begin{array}{c}
1 \\
0 \\
\vdots \\
0 \\
u_0
\end{array} \right) \).

For \( g = 0 \), \( F = 0 \),
\[ \frac{d}{dt} \hat{u} = -D_i \hat{u} - \frac{\sigma}{h} u_0 H^{-1} \hat{u}_0. \]

\[ D_i \text{ SBP wrt } H \quad \langle \hat{u}_i, D_i \hat{u}_i \rangle_{H^{-1}} = \frac{1}{2} (u_i^2 - u_{i+1}^2). \]
\[
\frac{d}{dt} \| \tilde{u} \|^2_{H^1} = 2 \langle \tilde{u}, \frac{d}{dt} \tilde{u} \rangle_{H^1}
\]

\[
= -2 \langle \tilde{u}, D \tilde{u} \rangle_{H^1} - 2 \sigma \| u_0 \|^2_{H^1} - 2 \sigma \langle \tilde{u}, H \tilde{e} \rangle_{H^1}
\]

\[
= - \frac{1}{N} (U_N^2 - U_0^2) - 2 \sigma \| u_0 \|^2_{H^1}
\]

\[
= - U_N^2 - (2 \sigma - 1) U_0^2.
\]

The solution is thus non-increasing in $H_1$ norm if $\sigma \geq \frac{1}{2}$, which is therefore the condition for stability.
c) The point-wise error at \( x_j = h_j \), \( j = 0, \ldots, N_j h + \frac{h}{N} \)
is 
\[ e_j = U(x_j, t) - U_j(t) \]

\[
\frac{d}{dt} e_j = \frac{\partial}{\partial x} U(x_j, t) - \frac{\partial}{\partial x} U_j(t) 
\]

\[ = -u_x + F(x_j, t) - \left( -D, u_j - 2 \frac{\sigma^2}{N} (u_0 - u(x_0, t)) \delta_j \right) 
\]

\[ = -u_x + D u_j + 2 \frac{\sigma^2}{N} (u_0 - u(x_0, t)) \delta_j 
\]

\[ = -u(x_j, t) + D (u_j - e_j) - 2 \frac{\sigma^2}{N} e_0 \delta_j 
\]

\[ = -D e_j - 2 \frac{\sigma^2}{N} e_0 \delta_j + \left( U(t) - U(t - h) \right) 
\]

\[
\frac{d^2 e}{dt^2} = -D \frac{d}{dx} \frac{\sigma^2}{N} \left( \begin{array}{c} e_j \\ \vdots \end{array} \right) + \mathcal{E}(t), \quad \text{truncation error}
\]

i.e., the equation for the error is identical to the semidiscretisation of the PDE. Given exact interpolation of the initial condition, \( e_0 = \mathcal{O}(h^2) \), (6) gives

\[ \| e_j(t) \|_{H^1_h} \leq \int_0^t \| \mathcal{E}(s) \|_{H^1_h} ds 
\]

\[ \leq \int_0^t \left( \frac{h^2}{2} C(h) + h \sum_{j=1}^{N-1} C(h_j) + \frac{h}{2} C(h) \right) ds 
\]

\[ \leq t \left( C(h^2) + C(h) + C(h) \right) 
\]

\[ \leq t \mathcal{O}(h^2), \quad \Rightarrow \text{second order convergence.} 
\]