1) $\vec{u}_t + A\vec{u}_x = \vec{0}$, $0 < x < 1$, $t > 0$

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

a) Determine the eigenvalues of A.

$\det(A - \lambda I) = 0$

$$0 = \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = (\lambda - 2)^2 - 9$$

$$\lambda = 2 \pm 3 \qquad \begin{cases} \lambda_1 = 5 \\ \lambda_2 = -1 \end{cases}$$

Since the eigenvalues are real and distinct, there exists an orthogonal transform which diagonalises A.

Let $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = V^{-1}AV$, $V^{-1} = V^T$,

and define the characteristic variables

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = V^{-1}\vec{u}.$$

Note that the difference operators D commute with non-singular matrices.

We have the difference scheme

$$\vec{u}_j^{n+1} = (I - kAD)\vec{u}_j^n.$$

Multiply from the left by V^{-1}, and insert the identity matrix at suitable places.

$$V^{-1}\vec{u}_j^{n+1} = V^{-1}(VV^{-1} - kAD)VV^{-1}\vec{u}_j^n$$

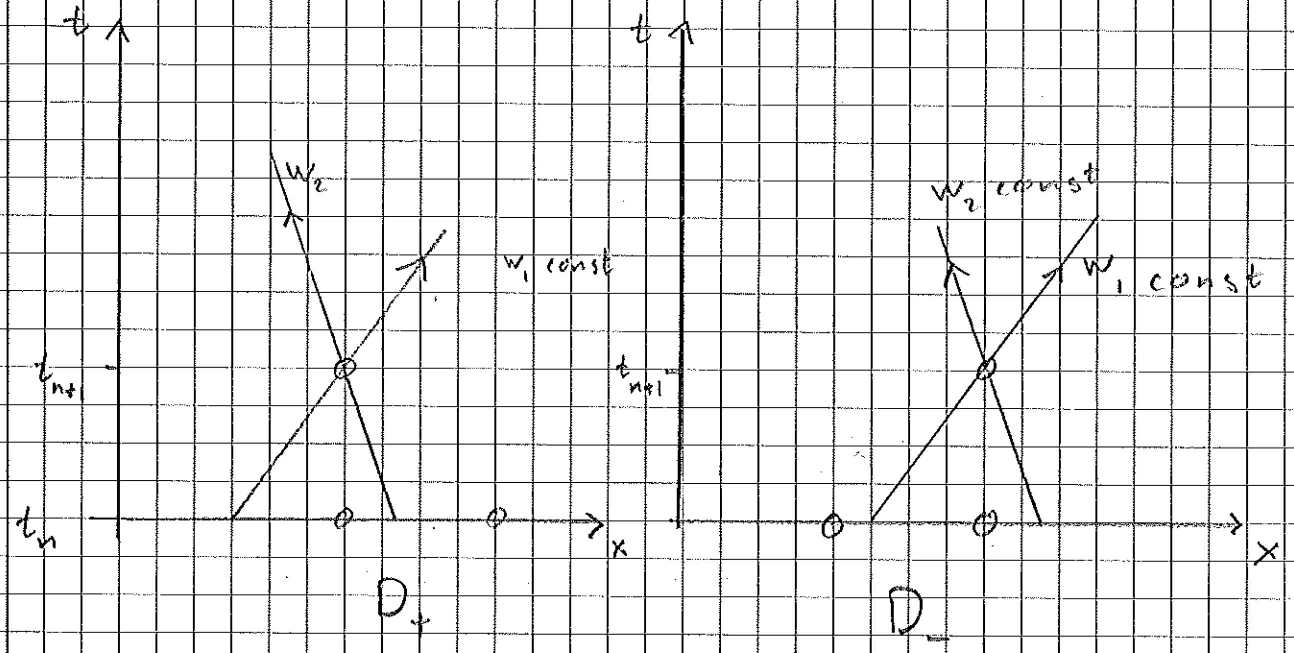
$$\vec{w}_j^{n+1} = (I - kV^{-1}ADV)\vec{w}_j^n$$

$$\vec{w}_j^{n+1} = (I - k\Lambda D)\vec{w}_j^n.$$

We have diagonalised the system and decoupled the equations.
We thus have two independent equations

$$(w_i)_j^{n+1} = (w_i)_j^n - k\lambda_i D(w_i)_j^n$$

The CFL-condition, that the numerical domain of dependence must include the analytical domain of dependence, is necessary for stability. We draw the stencils for D_+ and D_- in a characteristics plot,



We see that for D_+, the domain of dependence for w_1 will fall out of the numerical domain of dependence regardless of the time step. For D_-, the domain of dependence for w_2 will fall out of the numerical domain of dependence regardless of the time step.

Both D_+ and D_- thus yield unstable discretisations.

For D_0 it is possible to fulfill the CFL-condition. Since the condition is not sufficient, we need another method of analysis.

Express w_i through the inverse discrete Fourier transform,

$$(w_i)_j^n = \sum_{\omega} (\hat{w}_i)_{\omega}^n e^{i\omega x_j}.$$

Since the equation is linear we can treat each frequency independently.
Make the ansatz

$$(w_i)_j^n = \hat{w}_i^n e^{i\omega x_j}$$

and insert it into the scheme with $D = D_0$.

$$\hat{w}_i^{n+1} e^{i\omega x_j} = \hat{w}_i^n e^{i\omega x_j} - \frac{k}{2h}\lambda_i \hat{w}_i^n \left(e^{i\omega(x_j+h)} - e^{i\omega(x_j-h)}\right)$$

$$\hat{w}_i^{n+1} = \hat{w}_i^n - \frac{k}{2h}\lambda_i \hat{w}_i^n \left(e^{i\omega h} - e^{-i\omega h}\right)$$

$$\hat{w}_i^{n+1} = \hat{w}_i^n - \frac{ik}{h}\lambda_i \sin(\omega h)\hat{w}_i^n$$

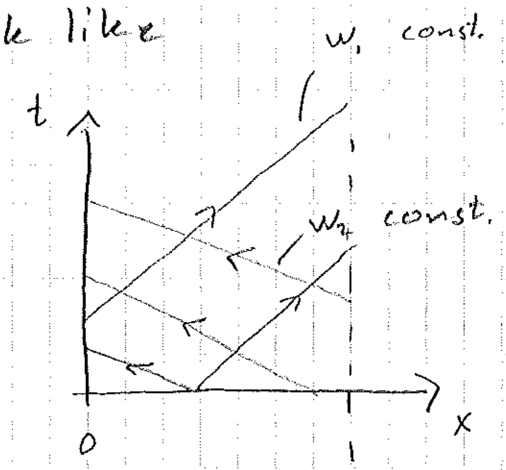
Stability if $|\hat{w}_i^{n+1}| \leq |\hat{w}_i^n| \Leftrightarrow$

$$\left|1 - \frac{ik}{h}\lambda_i \sin(\omega h)\right| \leq 1 \quad , \quad i = 1, 2$$

This is violated for all frequencies $\omega \neq 0$.

All three proposed schemes are thus unstable.

b) The characteristics look like



We need B.C. for W_1 at $x=0$ and for W_2 at $x=1$

$$\begin{cases} W_1 = \frac{1}{\sqrt{2}}(u_1 + u_2) \\ W_2 = \frac{1}{\sqrt{2}}(u_1 - u_2) \end{cases} \Leftrightarrow \begin{cases} u_1 = \frac{1}{\sqrt{2}}(W_1 + W_2) \\ u_2 = \frac{1}{\sqrt{2}}(W_2 - W_1) \end{cases}$$

(i) $u_1(0,t) = 0$
$W_1(0,t) = -W_2(0,t)$ - ok

$u_2(1,t) = 0$
$W_2(1,t) = W_1(1,t)$ - ok

$\Rightarrow$ well-posed.

(ii) Gives no boundary data at $x=1$ $\Rightarrow$ ill-posed.

(iii) $u_1(0,t) + u_2(0,t) = 0$
$W_1(0,t) = 0$ - ok

$u_1(1,t) + u_2(1,t) = 0$
$W_1(1,t) = 0$ — need for W_2
$\Rightarrow$ ill-posed.

2) $u_t = (a(x)u)_x$, $0 < x < 2\pi$, $t > 0$
periodic boundary conditions.

$h = \frac{2\pi}{N}$, $x_j = jh$, $j = 0, 1, \ldots, N-1$.

$u_j \sim u(x_j)$.

a) Let $v_j = a(x_j)u_j$,

$$\hat{v}_k = \sum_{j=0}^{N-1} v_j e^{-2\pi i jk/N} \qquad \text{(DFT)}$$

$$v_j = \frac{1}{N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \hat{v}_k e^{2\pi i jk/N}. \qquad \text{(IDFT)}$$

To evaluate $(a(x)u)_x$ approximately,

1) Calculate $v_j = a(x_j)u_j$

2) Apply DFT to get $\hat{v}_k$.

3) Differentiate in Fourier space,

$$\frac{\partial}{\partial x_j} v_j = \frac{\partial}{\partial x_j} \frac{1}{N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \hat{v}_k e^{ikx_j} = \frac{1}{N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} (ik\hat{v}_k) e^{ikx_j}$$

4) Apply IDFT to $ik\hat{v}_k$.

b) The discrete Fourier transform reads

$$a(y_j) = \sum_{k=0}^{N-1} a_k \, y_j^k, \quad j = 0, \ldots, N-1., \quad N = 2M$$

a_k are (typically) point-wise values of a function, and

$$y_j^k = e^{-2\pi i jk/N}$$

are the Fourier basis functions evaluated at point j. The sum is split as

$$a(y_j) = \underbrace{\sum_{k=0}^{M-1} a_{2k} (y_j^2)^k}_{(1)_j} + \underbrace{y_j \sum_{k=0}^{M-1} a_{2k+1} (y_j^2)^k}_{(2)_j}, \quad j = 0, \ldots, M-1.$$

With this expression we can evaluate

$$a(y_j) = (1)_j + (2)_j, \quad j = 0, \ldots, M-1.$$

By noting that

$$y_{j+M} = e^{-2\pi i j/N - 2\pi i \frac{M}{N}} = y_j e^{-\pi i} = -y_j$$

we can construct the rest of the Fourier coefficients,

$$a(y_j) = (1)_j - (2)_j, \quad j = M, \ldots, N-1.$$

3. a) Definition: A FD scheme for the conservation law

$$u_t + f(u)_x = 0$$

is <u>conservative</u> if it can be written as

$$u_j^{n+1} = u_j^n - \frac{k}{h}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right),$$

where F is Lipschitz continuous and $F(u,u,\cdots,u) = f(u)$.

This is fulfilled for the given scheme

$$u_j^{n+1} = u_j^n - \frac{k}{2h}\left(f_{j+1}^n - f_{j-1}^n\right) + \frac{k^2}{2h^2}\Big(D_{j+1}^{n\prime}\left(f_{j+1}^n - f_j^n\right) + -D_j^{n\prime}\left(f_j^n - f_{j-1}^n\right)\Big),$$

by

$$F_{j+\frac{1}{2}}^n = \frac{1}{2}\left(f_{j+1}^n + f_j^n\right) - \frac{k}{2h}\left(D_{j+1}^n - f_j^n\right).$$

b) $u_t + (f(u))_x = 0$, $f(u) = u^4$

$$u(x,0) = \begin{cases} u_L = 1, & x < 0 \\ u_R = -1, & x > 0 \end{cases}$$

Initially we have a shock at $x = 0$. The shock propagation speed is given by the Rankine-Hugoniot condition,

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{1^4 - (-1)^4}{1 - (-1)} = \frac{0}{2} = 0.$$

The entropy condition,

$$4 = 4 \cdot 1^3 = f'(u_L) > s > f'(u_R) = 4(-1)^3 = -4,$$

is fulfilled. The solution will therefore have a shock also for $t > 0$.

Wherever $u_x = 0$, i.e. away from $x = 0$, $u_t = 0$.

The solution to the Riemann problem is then

$$u(x,t) = u(x,0) = \begin{cases} 1, & x < 0 \\ -1, & x > 0. \end{cases}$$

c) The linearised equation is

$$u_t + 4u_x = 0. \qquad (*)$$

It has the analytical solution

$$u(x,t) = u(x-4t, 0),$$

i.e. waves travel to the right with speed 4 length units per time unit.

Denote the linearised flux function by $\tilde{f}(u) = 4u$. Let $\tilde{f}_j^n = \tilde{f}(u_j^n)$, and $\tilde{D}_j^n = \tilde{f}'(u_j^n)$. Insert this into the scheme,

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(\tilde{f}_{j+1}^n - \tilde{f}_{j-1}^n\right) +$$
$$+ \frac{\lambda^2}{2}\left(\tilde{D}_{j+1}^n\left(\tilde{f}_{j+1}^n - \tilde{f}_j^n\right) - \tilde{D}_j^n\left(\tilde{f}_j^n - \tilde{f}_{j-1}^n\right)\right)$$

$$= u_j^n - 2\lambda\left(u_{j+1}^n - u_{j-1}^n\right) +$$
$$+ 8\lambda^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right).$$

Make the ansatz

$$u_j^n = \hat{u}^n e^{i\omega x_j}.$$

The analytical solution to $(*)$ would then be

$$u(x_j, t) = \hat{u}\exp(i\omega(x_j - 4t)).$$

Inserting the ansatz yields

$$\hat{u}^{n+1} e^{i\omega x_j} = \hat{u}^n e^{i\omega x_j} - 2\lambda \hat{u}^n \left(e^{i\omega(x_j+h)} - e^{i\omega(x_j-h)} \right) +$$
$$+ 8\lambda^2 \hat{u}^n \left(e^{i\omega(x_j+h)} - 2e^{i\omega x_j} + e^{i\omega(x_j-h)} \right)$$

$$\hat{u}^{n+1} = \left(1 - 2\lambda \left(e^{i\omega h} - e^{-i\omega h} \right) + 8\lambda^2 \left(e^{i\omega h/2} - e^{-i\omega h/2} \right)^2 \right) \hat{u}^n$$

$$\hat{u}^{n+1} = \underbrace{\left(1 - 4i\lambda \sin(\omega h) - 32\lambda^2 \sin^2\left(\frac{\omega h}{2}\right) \right)}_{g(\omega h)} \hat{u}^n$$

$$\hat{u}^{n+1} = |g| \exp(i \operatorname{Arg}(g)) \hat{u}^n$$

$$u_j^{n+1} = \hat{u}^{n+1} e^{i\omega x_j}$$
$$= |g| \hat{u}^n \exp(i\omega x_j + i \operatorname{Arg}(g))$$
$$= |g| \hat{u}^n \exp\left(i\omega \left(x_j + \frac{\operatorname{Arg}(g)}{\omega}\right)\right)$$

The correct solution at t_{n+1}, given an exact u_j^n, would be

$$u(x_j, t_{n+1}) = \hat{u}^n e^{i\omega(x_j - 4k)}$$

Thus, the deviation of $|g|$ from 1 is the damping error, and the deviation of $\frac{\operatorname{Arg} g}{\omega k}$ from -4 is the dispersion error.

$$\frac{\operatorname{Arg} g}{\omega k} = \frac{1}{\omega k} \operatorname{Arctan}\left(\frac{-4\lambda \sin(\omega h)}{1 - 32\lambda^2 \sin^2\left(\frac{\omega h}{2}\right)}\right)$$

Asymptotically, as $\omega h \to 0$

$$\lim_{\omega h \to 0} \frac{\operatorname{Arg} g}{\omega k} =$$

$$= \lim_{\omega h \to 0} \frac{1}{\omega k} \arctan\left(\frac{-4\lambda \sin(\omega h)}{1 - 32\lambda^2 \sin^2\left(\frac{\omega h}{2}\right)}\right)$$

$$= \lim_{\omega h \to 0} \frac{1}{\omega k} \arctan\left(-4\lambda \sin(\omega h)\right)$$

$$= \lim_{\omega h \to 0} \frac{1}{\omega k} (-4) \frac{k}{h} \sin(\omega h)$$

$$= -4 \lim_{\omega h \to 0} \frac{\sin(\omega h)}{\omega h} = -4,$$

i.e. we get the correct wave speed.

As $\omega h \to \pi$, $g(\omega h)$ gets real, and assuming $32\lambda^2 \leq 1$, which is necessary for stability, $\arg g \to 0$. Therefore, the highest frequency representable gets the wave speed 0.

4) $\begin{cases} u_t = u_{xx}, & 0<x<1,\ t>0 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x) \end{cases}$

a) $\frac{d}{dt}\|u\|^2 = 2(u, u_t) = 2(u, u_{xx}) =$

$= 2uu_x\Big|_0^1 - 2(u_x, u_x) = -2\|u_x\|^2 \leq 0.$ $\square$

$\Rightarrow \|u(\cdot, t)\| \leq \|f\|.$

b) $\sum_{j=1}^{N} v_j D_+ w_j + \sum_{j=1}^{N} w_j D_- v_j =$

$= \frac{1}{h}\sum_{j=1}^{N}\left(v_j w_{j+1} - v_j w_j + w_j v_j - w_j v_{j-1}\right)$

$= \frac{1}{h}\left(\sum_{j=1}^{N-1} v_j w_{j+1} + v_N w_{N+1} - v_0 w_1 - \sum_{j=1}^{N-1} v_j w_{j+1}\right)$

$= -\frac{1}{h} v_0 w_1 + \frac{1}{h} v_N w_N.$ $\square$

c) Introduce the grid function $u_j(t)$ on the same mesh. $u_0(t) = u_{N+1}(t) = 0$ through injection. Spatial derivatives by D_+D_-. The semidescritisation then reads,

$$\begin{cases} \frac{du_j}{dt} = D_+ D_- u_j \quad , \; j = 1, \ldots, N \\ u_0 = u_{N+1} = 0 \end{cases}$$

$$\frac{d}{dt}\|\vec{u}\|^2 = 2\langle \vec{u}, \dot{\vec{u}} \rangle_h = 2\langle \vec{u}, D_+ D_- \vec{u} \rangle_h$$

$\{$Let $v_j = D_- u_j\}$

$$= 2\sum_{j=1}^{N} u_j D_+ D_- u_j = \cancel{-\frac{2}{h} u_0 v_1}^{\to 0} - 2\sum_{j=1}^{N} v_j \underbrace{D_- u_j}_{v_j} + \frac{2}{h} u_N v_{N+1}$$

$$= -\frac{2}{h^2}\sum_{j=1}^{N}(u_j - u_{j-1})^2 + \frac{2}{h^2} u_N (\cancel{u_{N+1}}^{\to 0} - u_N)$$

$$= -\frac{2}{h^2}\sum_{j=1}^{N}(u_j - u_{j-1})^2 - \frac{2}{h^2} u_N^2 \le 0 \quad \square.$$

d) The semi-discretisation can be written

$$\frac{d}{dt}\tilde{U}(t) = A\tilde{U}(t), \quad \text{where } \tilde{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}.$$

Since A is symmetric its eigenvalues are real, and the stability estimate derived in c) shows that A is negative definite. Thus, all its eigenvalues lie on the negative real axis, which is in the stability domain of the integrator. The method is thus stable.

5) $\begin{cases} u_t + u_x = F(x,t), & 0 < x < 1,\ t > 0 \\ u(x,0) = f(x) \\ u(0,t) = g(t) \end{cases}$

$\vec{u}(t) = \{u_i(t)\}$ grid function at $x_i = ih$, $i = 0, \dots, N$.

D_1 operator s.t. $D_1 v_j = \begin{cases} D_+ v_j, & j = 0 \\ D_0 v_j, & j = 1, \dots, N-1 \\ D_- v_j, & j = N. \end{cases}$

Corresponding inner product and norm given by the diagonal matrix H,

$$\langle \vec{u}, \vec{v} \rangle_{H,h} = \langle \vec{u}, H\vec{v} \rangle_h = \frac{h}{2} u_0 v_0 + \sum_{i=1}^{N-1} h u_i v_i + \frac{h}{2} u_N v_N.$$

a) Discretise the PDE in space by

$$\frac{d}{dt}\vec{u} = -D_1 \vec{u} - \frac{\sigma}{h} H^{-1}(u_0 - g(t))\vec{e}_0 + \vec{F}(t),$$

where $\vec{e}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

For $g = 0$, $F = 0$,

$$\frac{d}{dt}\vec{u} = -D_1 \vec{u} - \frac{\sigma}{h} u_0 H^{-1} \vec{e}_0$$

D_1 SBP w.r.t. $H \iff \langle \vec{u}, D_1 \vec{u} \rangle_{H,h} = \frac{1}{2}(u_N^2 - u_0^2)$.

$$\frac{d}{dt}\|\vec{u}\|^2_{H,h} = 2\langle \vec{u}, \frac{d}{dt}\vec{u}\rangle_{H,h}$$

$$= -2\langle \vec{u}, D\vec{u}\rangle_{H,h} - 2\frac{\sigma}{h}u_0\langle \vec{u}, H^{-1}\vec{e}_0\rangle_{H,h}$$

$$= -(u_N^2 - u_0^2) - 2\frac{\sigma}{h}u_0 h u_0$$

$$= -u_N^2 - (2\sigma - 1)u_0^2.$$

The solution is thus non-increasing in H-norm if $\sigma \geq \frac{1}{2}$, which is therefore the condition for stability.

c) The point-wise error at $x_j = hj$, $j = 0, \dots, N$, $h = \frac{1}{N}$ is $e_j = u(x_j, t) - u_j(t)$.

$$\frac{d}{dt} e_j = \frac{\partial}{\partial t} u(x_j, t) - \frac{d}{dt} u_j(t)$$

$$= -u_x + F(x_j, t) - \left(-D_1 u_j - 2\frac{\sigma}{h}(u_0 - g)\delta_{j0} + F_j(t)\right)$$

$$= -u_x + D_1 u_j + 2\frac{\sigma}{h}(u_0 - u(x_0, t))\delta_{j0}$$

$$= -u_x + D_1(u - e_j) - 2\frac{\sigma}{h} e_0 \delta_{j0}$$

$$= -u_x + u_x + \tau_j - D_1 e_j - 2\frac{\sigma}{h} e_0 \delta_{j0}$$

$$= -D_1 e_j - 2\frac{\sigma}{h} e_0 \delta_{j0} + \tau_j \qquad \tau_j = \begin{cases} O(h), & j = 0, N \\ O(h^2), & \text{else} \end{cases}$$

$$\frac{d}{dt}\vec{e} = -D_1 \vec{e} - \frac{\sigma}{h} H^{-1}\begin{pmatrix} e_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \vec{\tau}(t),$$

truncation error

i.e. the equation for the error is identical to the semidiscretisation of the PDE. Given exact interpolation of the initial condition, $\vec{e}(0) = \vec{0}$, (6) gives

$$\|\vec{e}(t)\|_{H,h} \leq \int_0^t \|\vec{\tau}(s)\|_{H,h}$$

$$\leq \int_0^t \left(\frac{h}{2} O(h) + h \sum_{j=1}^{N-1} O(h^2) + \frac{h}{2} O(h)\right) ds$$

$$\leq t\left(O(h^2) + O(h^2) + O(h^2)\right)$$

$$\leq t\, O(h^2). \quad \Rightarrow \text{second order convergence.}$$