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Solution Suggestion to the Examination in Analysis of Numerical Methods, 2001-10-12

1. Show consistency and stability. Then, the Lax-Richtmyer theorem yields that the scheme is convergent. Taylor expand the operator D_+D_-

$$D_+D_-u = \frac{\partial^2 u}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} + O(h^4) = \left(1 + \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 u}{\partial x^2} + O(h^4) = \left(1 + \frac{h^2}{12}D_+D_-\right)\frac{\partial^2 u}{\partial x^2} + O(h^4)$$

That gives

$$\left(1 + \frac{h^2}{12}D_+D_-\right)\cdot\left(\frac{u_j^{n+1} - u_j^n}{k} - \lambda\frac{\partial^2 u}{\partial x^2} - f_j\right) = O(h^4)$$

Moreover, we have that

$$\frac{u_j^{n+1} - u_j^n}{k} = u_t + O(k)$$

which yields when inserted into the equation above that the scheme is consistent.

The solution can be split into $u(x,t) = u_h(x,t) + u_p(x)$, where $u_h(x,t)$ is the solution to the problem with homogeneous right hand side and $u_p(x)$ is a time independent particular solution, i.e. a solution to $-\lambda u_{xx} = f(x)$ and does not affect stability. Investigate the stability for the homogeneous problem with the Fourier method. Ansatz $u_j^n = g^n e^{i\omega x}$ and using that

$$D_+D_-e^{i\omega x} = -\frac{4}{h^2}\sin^2(\frac{\omega h}{2})e^{i\omega x}$$

Inserting and factorizing gives

$$\left(1 - \frac{4h^2}{12h^2}\sin^2(\frac{\omega h}{2})\right)\left(\frac{g-1}{k}\right) + \frac{4\lambda}{h^2}\sin^2(\frac{\omega h}{2}) = 0$$

$$\Rightarrow$$

$$g = 1 - \frac{4\frac{\lambda k}{h^2}\sin^2(\frac{\omega h}{2})}{1 - \frac{1}{3}\sin^2(\frac{\omega h}{2})}$$

For stability we require $|g| \leq 1$. We have

$$1 \geq 1 - \frac{4\frac{\lambda k}{h^2}\sin^2(\frac{\omega h}{2})}{1 - \frac{1}{3}\sin^2(\frac{\omega h}{2})} \geq 1 - 6\frac{\lambda k}{h^2} \geq -1 \quad \text{for} \quad \frac{\lambda k}{h^2} \leq \frac{1}{3}$$

Therefore, the method is consistent and stable and thus convergent under the given condition.

2. We have variable coefficients, but $a(x)$ is Lipschitz continuous. Freeze $a(x) = a$ constant. Divide the problem into two quarter plane problems, $0 \leq x < \infty$ and $-\infty < x \leq 1$. Set $f(x,t) = h(x) = g(t) = 0$ and investigate each quarter plane problem for itself.

Check first the basic condition, i.e. that the leapfrog scheme without boundary conditions is stable. Fourier analysis yields that

$$|a\lambda| < 1$$

That should be valid for all values which a can have in the interval. The maximum for $a = \pi^2$ gives $k/h < 1/\pi^2$ as stability condition. Start now with the GKSO-analysis.

Step 1: *Resolvent equation*
Ansatz $v_i^n = z^n\tilde{v}_i$

$$\Rightarrow z^{n+1}\tilde{v}_i = z^{n-1}\tilde{v}_i + a\lambda z^n(\tilde{v}_{i+1} - \tilde{v}_{i-1})$$

$$z^2\tilde{v}_i = \tilde{v}_i + a\lambda z(\tilde{v}_{i+1} - \tilde{v}_{i-1})$$

Step 2. *Characteristic equation*
Ansatz $\tilde{v}_i = \kappa^i$

$$\Rightarrow z^2\kappa^i = \kappa^i + a\lambda z(\kappa^{i+1} - \kappa^{i-1})$$

$$(z^2 - 1)\kappa = a\lambda z(\kappa^2 - 1)$$

$$\kappa^2 - \frac{z^2 - 1}{a\lambda z}\kappa - 1 = 0$$

Step 3. *Determinant condition*
The characteristic equation yields that its roots satisfy

$$\kappa_1 \cdot \kappa_2 = -1 \quad \Rightarrow \quad |\kappa_1| < 1,\ |\kappa_2| > 1 \text{ or } |\kappa_1| = |\kappa_2| = 1$$

Check $\kappa = e^{i\theta}$ (Fourier ansatz) $\Rightarrow |z| \leq 1$ because of the basic assumption of stability for periodic boundary conditions. Thus, for $|z| > 1$ we have $|\kappa_1| < 1$ and $|\kappa_2| > 1$. We can write the solution as $\tilde{v}_j = \sigma_1\kappa_1^j + \sigma_2\kappa_2^j$, but $|\kappa_2| > 1$ and $v_j^n \in l_2(0, \infty) \Rightarrow \sigma_2 = 0$ constraint. Look for a solution in the form $\tilde{v}_j = \sigma\kappa^j$ where $|\kappa| < 1$. Insert the ansatz $v_j^n = z^n\sigma\kappa^j$ into the boundary condition.
For the right boundary condition, we get

$$z^n\sigma\kappa^j = 0$$

which does not give any non-trivial solutions and therefore does not affect stability.
The left boundary condition yields

$$z^{n+1}\sigma(1 + a\lambda) = a\lambda z^{n+1}\sigma\kappa + z^n\sigma$$

$$(z - 1 - a\lambda z(\kappa - 1))\sigma = 0$$

We look for non-trivial solutions with $\sigma \neq 0$, i.e.

$$z - 1 - a\lambda z(\kappa - 1) = 0$$

Step 4. *Solve the equations*

$$\begin{cases} (z^2 - 1)\kappa = a\lambda z(\kappa^2 - 1) \\ z - 1 - a\lambda z(\kappa - 1) = 0 \end{cases}$$

$$\Rightarrow$$

$$\kappa = 1, z = 1$$

Step 5. *Check the solution*
Is the solution obtained in the limit $|z| \to 1_+$? The characteristic equation yields that $\kappa_1 = 1$ and $\kappa_2 = -1$ or $\kappa_1 = -1$ and $\kappa_2 = 1$. Only the first case is a solution (as $\sigma_2 = 0$). We know that $|\kappa_1| < 1$ and $|\kappa_2| > 1$ for $|z| > 1$. Make the ansatz $z = 1 + \delta$ and $\kappa = 1 + \varepsilon$ with $\delta > 0$ and check the sign of ε. Insert the ansatz into the characteristic equation and neglect higher order terms,

$$((1 + \delta)^2 - 1)(1 + \varepsilon) = a\lambda(1 + \delta)((1 + \varepsilon)^2 - 1)$$

$$\Rightarrow$$

$$2\delta \approx a\lambda 2\varepsilon$$

Thus, $\varepsilon > 0$ and it is κ_2 which satisfies the equations. We have no solution for $|z| > 1$ or when $|z| \to 1_+$.

Step 6. *Conclusions*
The leapfrog scheme is stable with the proposed boundary condition, if $k/h < 1/\pi^2$.

3. The operator $D_+D_-(I - \frac{h^2}{12}D_+D_-)$ is five points wide. Boundary conditions are needed in all points, which are one point away from the boundary. Taylor expand $u(x,y)$ normal to the boundary.

   For the boundary $x = h_1$, we obtain

$$u(h_1,y) = u(0,y) + h_1 u_x(0,y) + \frac{h_1^2}{2}u_{xx}(0,y) + \frac{h_1^3}{6}u_{xxx}(0,y) + O(h_1^4)$$

   Use that $u(x,y) = g(x,y)$ on the boundary. As boundary condition, we get then

$$u(h_1,y) = g(0,y) + h_1 g_x(0,y) + \frac{h_1^2}{2}g_{xx}(0,y) + \frac{h_1^3}{6}g_{xxx}(0,y)$$

   In the same way, we get for the other boundaries the boundary conditions

$$\begin{aligned}
u(1-h_1,y) &= g(1,y) - h_1 g_x(1,y) + \frac{h_1^2}{2}g_{xx}(1,y) - \frac{h_1^3}{6}g_{xxx}(1,y)\\
u(x,h_2) &= g(x,0) + h_2 g_y(x,0) + \frac{h_2^2}{2}g_{yy}(x,0) + \frac{h_2^3}{6}g_{yyy}(x,0)\\
u(x,1-h_2) &= g(x,1) - h_2 g_y(x,1) + \frac{h_2^2}{2}g_{yy}(x,1) - \frac{h_2^3}{6}g_{yyy}(x,1)
\end{aligned}$$

4. Ansatz $u = U + v$ where U is a solution to the differential equation and v a small perturbation. Study how the small perturbation affects the solution. I.e. check the stability for v. Insert $u = U + v$ into the equation and neglect higher order terms in v (i.e. linearize the equation). Express the differential equation as $u_t + f'(u)u_x = 0$ and exploit that U is a solution. That yields

$$v_t + f'(U)v_x = -f''(U)U_x v$$

   Strang's theorem states that if u is sufficiently smooth and the difference method D is stable for the linearized equation, then D also converges for the nonlinear equation.
   Desirable properties of a numerical scheme for nonlinear conservation laws:

   a) *Conservative form:* A scheme for the conservation law $u_t + f(u)_x = 0$ is in conservative form, if it can be written as

$$\frac{u_j^{n+1} - u_j^n}{k} + \frac{h_{j+1/2}^n - h_{j-1/2}^n}{h} = 0$$

   where $h_{j+1/2}^n = h(u_{j-q}^n, ..., u_{j+p}^n)$. For consistency, we require (as $h \to 0$, $k \to 0$) that $h(u,...,u) = f(u)$. The conservative form guarantees that possible shocks are computed with the correct shock speed.

   b) *Entropy satisfying:* Gives the physically correct solution, breaks up an expansion shock into an expansion wave.

   c) *TVD:* A scheme is called total variation diminishing (TVD), if

$$\sum_j |v_j^{n+1} - v_{j-1}^{n+1}| \le \sum_j |v_j^n - v_{j-1}^n|$$

   Prevents growth of total variation in the solution, namely through high wave number oscillations.

5. The error can be expressed as a linear combination of the eigenvectors of A. On level l, we have the wave numbers

$$v_\mu^l = \sqrt{2h_l}\sin(\mu\pi x) \quad \mu = 1, \dots, n_l$$

   and on level $l-1$

$$v_\mu^{l-1} = \sqrt{2h_{l-1}}\sin(\mu\pi x) \quad \mu = 1, \dots, n_{l-1}$$

   where $n_l = 2n_{l-1} + 1$ and $h_l = h_{l-1}/2$.
   Thus, the wave numbers $\ge n_{l-1} + 1 = \frac{n_l+1}{2}$ are only represented on the fine grid.

   Apply the restriction operator to the eigenvectors and study what happens componentwise

$$\left[Rv_\mu^l\right]_i = \frac{\sqrt{2h_l}}{4}(\sin(\mu\pi h_l(2i-1)) + 2\sin(\mu\pi h_l 2i) + \sin(\mu\pi h_l(2i+1))) = \dots =$$

$$= \frac{\sqrt{2h_l}}{4} 2(\cos(\mu\pi h_l) + 1)\sin(\mu\pi h_l 2i) = \sqrt{2h_l}\cos^2(\frac{\mu\pi h_l}{2})\sin(\mu\pi h_{l-1} i)$$

For $\mu \geq \frac{n_l+1}{2}$, express $\mu = n_l - k + 1$ where $k = 1, \dots, n_{l-1}$. That gives

$$\sin((n_l - k + 1)\pi h_{l-1} i) = \sin((2(n_{l-1}+1) - k)\pi h_{l-1} i) = \sin(2\pi i - k\pi h_{l-1} i) = -\sin(k\pi h_{l-1} i)$$

and we get

$$\left[Rv_\mu^l\right]_i = -\frac{1}{\sqrt{2}}\cos^2(\frac{\mu\pi h_l}{2})\left[v_k^{l-1}\right]_i$$

I.e. the wave number $\mu = n_l - k + 1$ on the fine grid is superimposed as wave number k on the coarse grid and damped by the factor $\frac{1}{\sqrt{2}}\cos^2(\frac{\mu\pi h_l}{2})$. The same analysis for the direct injection yields

$$\left[Rv_\mu^l\right]_i = \sqrt{2h_l}\sin(\mu\pi h_l 2i) = \sqrt{2h_l}\sin(\mu\pi h_{l-1} i) =$$

$$[\mu = n_l - k + 1 = 2(n_{l-1}+1) - k]$$

$$= \sqrt{2h_l}\sin(2\pi i - k\pi h_{l-1} i) = -\frac{1}{\sqrt{2}}\left[v_k^{l-1}\right]_i$$

Again, the wave number $\mu = n_l - k + 1$ on the fine grid is superimposed as wave number k on the coarse grid and damped by the factor $1/\sqrt{2}$ (less damping than for full weighting).