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Solutions to the Examination in Analysis of Numerical Methods

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1. a) Taylor expansion of the exact solution $u(x,t)$ around (x_j, t^n) yields:

$$\begin{aligned} u(x_j, t_{n\pm1}) &= u \pm ku_t + k^2/2\, u_{tt} + \pm k^3/6\, u_{ttt} + \mathcal{O}(k^4) \\ u(x_{j\pm1}, t_n) &= u \pm hu_x + h^2/2\, u_{xx} \pm h^3/6\, u_{xxx} + \mathcal{O}(h^4) \end{aligned}$$

Inserting $u = u(x_j, t_n)$ for v_j^n into the left hand side of the FDM

$$P_{k,h}v_j^n = \frac{v_j^{n+1} - v_j^{n-1}}{2k} - b\frac{v_{j+1}^n - v_j^{n+1} - v_j^{n-1} + v_{j-1}^n}{h^2} = 0 \tag{1}$$

and using the PDE $u_t - bu_{xx} = 0$ yields the truncation error

$$\begin{aligned} P_{k,h}u &= u_t + k^2/6\, u_{ttt} + \mathcal{O}(k^4) - b(u_{xx} + h^2/12\, u_{xxxx} + \mathcal{O}(h^4) - k^2/h^2\, u_{tt} + \mathcal{O}(k^4/h^2)) \\ &= k^2/6\, u_{ttt} + \mathcal{O}(k^4) - b(h^2/12\, u_{xxxx} + \mathcal{O}(h^4) - k^2/h^2\, u_{tt} + \mathcal{O}(k^4/h^2)) \\ &= \mathcal{O}(k^2) + \mathcal{O}(h^2) + bk^2/h^2\, u_{tt} + \mathcal{O}(k^4/h^2) \end{aligned}$$

If $\mu = \frac{k}{h^2}$ is constant, then $k^2 = \mu^2 h^4$ and $bk^2/h^2\, u_{tt} = \mathcal{O}(h^2)$. Thus, for $\mu = \frac{k}{h^2}$ constant, the FDM is second order accurate in space and time, i.e. the FDM is consistent. In fact, $P_{k,h}u = \mathcal{O}(h^2)$.
If $\lambda = \frac{k}{h}$ is constant, then $k^2 = \lambda^2 h^2$ and $P_{k,h}u = u_t - bu_{xx} + b\lambda^2\, u_{tt} + \mathcal{O}(k^2) + \mathcal{O}(h^2)$. Thus, for $\lambda = \frac{k}{h}$ constant, the FDM is consistent with the PDE $u_t - bu_{xx} + b\lambda^2\, u_{tt} = 0$, but not with the heat equation $u_t - bu_{xx} = 0$!

b) We perform the Fourier analysis by inserting the ansatz

$$v_j^n = g^n e^{i\omega x_j}$$

into the FDM (1) multiplied by $2k$. We get after dividing by $g^{n-1}e^{i\omega x_j}$

$$g^2 - 1 = d(ge^{i\omega h} - g^2 - 1 + ge^{-i\omega h})$$

where $d = 2bk/h^2$. Using $e^{i\omega h} + e^{-i\omega h} = 2cos(\omega h)$ and ordering the terms, we get the second order polynomial for the amplification factor g:

$$g^2 - \frac{2d\, cos(\omega h)}{1+d}g - \frac{1-d}{1+d} = 0\ . \tag{2}$$

The solutions are

$$g_{1/2} = \frac{d\, cos(\omega h)}{1+d} \pm \sqrt{\frac{d^2 cos^2(\omega h)}{(1+d)^2} + \frac{1-d}{1+d}}$$

or

$$g_{1/2} = \frac{d\, cos(\omega h) \pm \sqrt{1 - d^2(1 - cos^2(\omega h))}}{1+d} = \frac{d\, cos(\omega h) \pm \sqrt{1 - d^2 sin^2(\omega h)}}{1+d}$$

We have to check the sign of the discriminant.

(a) $1 - d^2 sin^2(\omega h) > 0$
Then the square root is real, and we can estimate using $d > 0$, $|cos(\omega h)| \leq 1$ and $-d^2 sin^2(\omega h) \leq 0$

$$|g_{1/2}| \leq \frac{d|cos(\omega h)| + \sqrt{1 - d^2 sin^2(\omega h)}}{1+d} \leq \frac{d+1}{1+d} = 1$$

Thus, the von Neumann condition $|g_{1/2}| \leq 1$ for simple roots is satisfied for this case.

(b) $1 - d^2 sin^2(\omega h) = 0$
Then we have a double root $g_1 = g_2$, which can be estimated as

$$|g_{1/2}| = \frac{d|cos(\omega h)|}{1+d} \leq \frac{d}{1+d} < 1$$

Thus, the von Neumann condition $|g_{1/2}| < 1$ for double roots is satisfied for this case.

(c) $1 - d^2 sin^2(\omega h) < 0$
Then the square root is imaginary, because $\sqrt{1 - d^2 sin^2(\omega h)} = i\sqrt{d^2 sin^2(\omega h) - 1}$. We get

$$|g_{1/2}|^2 = \frac{d^2 cos^2(\omega h) + d^2 sin^2(\omega h) - 1}{(1+d)^2} = \frac{d^2 - 1}{(1+d)^2} < 1$$

Thus, the von Neumann condition $|g_{1/2}| \leq 1$ for simple roots is also satisfied for this case.

As the von Neumann condition is satisfied for all possible roots of (2), the FDM (1) is unconditionally stable.

2. We perform the GKS analysis. The stability of FDM (1) for periodic boundary conditions was shown in task 1.b).

1. Resolvent equation

Inserting the ansatz $v_j^n = z^n \tilde{v}_j$ into the FDM (1) yields after dividing by z^{n-1}

$$(z^2 - 1)\tilde{v}_j - d\left[z(\tilde{v}_{j+1} + \tilde{v}_{j-1}) - (z^2+1)\tilde{v}_j\right] = 0\,, \tag{3}$$

which is the resolvent equation.

2. Characteristic equation

Inserting the ansatz $\tilde{v}_j = \kappa^j$ into the resolvent equation (3), we obtain

$$(z^2 - 1)\kappa^j - d\left[z(\kappa^{j+1} + \kappa^{j-1}) - (z^2+1)\kappa^j\right] = 0\,.$$

Dividing by κ^{j-1}, we get the characteristic equation

$$(z^2 - 1)\kappa - d\left[z(\kappa^2 + 1) - (z^2+1)\kappa\right] = 0\,. \tag{4}$$

Ordering the terms, (4) can also be expressed as

$$\kappa^2 - \frac{z^2 - 1 + d(z^2+1)}{dz}\kappa + 1 = 0\,. \tag{5}$$

3. Determinant condition

The general solution of the resolvent equation (3) is

$$\tilde{v}_j = \begin{cases} \sigma_1 \kappa_1^j + \sigma_2 \kappa_2^j & \text{if } \kappa_1 \neq \kappa_2 \\ \sigma_1 \kappa^j + \sigma_2 j \kappa^{j-1} & \text{if } \kappa_1 = \kappa_2 = \kappa \end{cases}$$

where κ_1 and κ_2 are the two roots of the characteristic equation (4). The coefficients σ_1 and σ_2 are determined such that the boundary condition $\frac{-3v_0^{n+1}+4v_1^{n+1}-v_2^{n+1}}{2h} = 0$ and $v^n \in l_2(0,\infty)$ are satisfied for the right quarter plane problem. (Similar reasoning is used for the left quarter plane problem with the boundary condition $v_N^{n+1} = 0$, cf. below.) Because of the form of the characteristic equation (5), the product of its roots satisfies the relation

$$\kappa_1 \kappa_2 = 1 \; .$$

Thus, either $|\kappa_1| < 1$ and $|\kappa_2| > 1$ or $|\kappa_1| = |\kappa_2| = 1$. In either case, we have to set $\sigma_2 = 0$ to secure $v^n \in l_2(0,\infty)$. Now, we check $|\kappa_1| = 1$. Inserting $\kappa = e^{i\xi}$ into the characteristic equation (4), we get for z the same result as for the amplification factor g in the von Neumann stability analysis (cf. task 1.b)), i.e. $|z| \leq 1$.
Thus, there can only be solutions with $|z| > 1$ for $|\kappa_1| < 1$ and $|\kappa_2| > 1$, and the solution must be of the form

$$v_j^n = z^n \sigma_1 \kappa_1^j \; .$$

(Inserting the solution into the boundary condition $v_N^{n+1} = 0$ yields $z^{n+1}\sigma_1\kappa_1^N = 0$. Thus, $\kappa_1 = 0$ for $\sigma_1 \neq 0$, and consequently $v_j^n = 0$. Therefore, the left quarter plane problem is stable.)
Inserting the solution into the boundary condition $\frac{-3v_0^{n+1}+4v_1^{n+1}-v_2^{n+1}}{2h} = 0$, we get

$$(-3 + 4\kappa_1 - \kappa_1^2)\sigma_1 = 0 \; .$$

For $\sigma_1 \neq 0$, we obtain the determinant condition

$$\kappa_1^2 - 4\kappa_1 + 3 = 0 \; . \tag{6}$$

4. Solve equations

The determinant condition (6) has the solutions $\kappa_1 = 1$ and $\kappa_1 = 3$. $\kappa_1 = 3$ can be excluded, because $|\kappa_1| < 1$, cf. above. However, $\kappa_1 = 1$ could be a solution in the limit $\kappa_1 \to 1_-$ as $|z| \to 1_+$.
We solve the characteristic equation (4) by inserting $\kappa = 1$ into (4). We get after ordering

$$z^2 - \frac{2d}{1+d} z - \frac{1-d}{1+d} = 0 \; .$$

Thus, we have the solutions $z = 1$ and $z = \frac{d-1}{1+d}$. As $|\frac{d-1}{1+d}| < 1$, the latter solution is uncritical and can be excluded.
Thus, the solution is

$$\kappa_1 = 1 \quad , \; z = 1 \; . \tag{7}$$

5. Check solutions

If $\kappa = 1$ and $z = 1$ were a solution in the limit $\kappa \longrightarrow 1_-$ and $z \longrightarrow 1_+$, the scheme would be unstable. To check that case, we assume that $z = 1 + \delta$ with $\delta > 0$ and $\kappa = 1 + \epsilon$. We insert z and κ into the characteristic equation (4) and check the sign of ϵ as $\delta \longrightarrow 0$. If

$\epsilon < 0$, the scheme is unstable.
Inserting z and κ into (4), yields

$$((1+\delta)^2 - 1)(1+\epsilon) - d\left[(1+\delta)((1+\epsilon)^2+1) - ((1+\delta)^2+1)(1+\epsilon)\right] \;=\; 0$$

Neglecting the third order terms, we get

$$2\delta + 2\delta\epsilon - d\left[\epsilon^2 - \delta^2\right] \;=\; 0\,. \tag{8}$$

Neglecting the second order terms yields $2\delta = 0$. Thus, $2\delta\epsilon - d\left[\epsilon^2 - \delta^2\right] = 0$, as $\delta \to 0$. We get $\epsilon(2\delta - d\epsilon)/d = \delta^2$. Therefore, $(2\delta - d\epsilon)/d \to 0$ as $\delta \to 0$. Thus, $\epsilon = 2\delta/d$. Consequently, $\epsilon > 0$, because $\delta > 0$ and $d > 0$. As $\epsilon > 0$, the scheme is stable, as discussed above.

6. Conclusions

The FDM (1) with the boundary conditions stated above is unconditionally stable.

3. a) The 1D Maxwell equations in a nonconducting medium can be expressed as

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = 0\,, \tag{9}$$

where

$$\mathbf{U} = \begin{pmatrix} B \\ E \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix}.$$

Determine eigenvalues λ of $\mathbf{A}$:

$$det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - c^2 = 0 \Longleftrightarrow \lambda = \pm c\,.$$

Thus, the eigenvalues $\lambda_1 = -c$ and $\lambda_2 = +c$ are real. Determine corresponding eigenvectors $\mathbf{r}_1$ and $\mathbf{r}_2$, i.e. $(\mathbf{A} - \lambda\mathbf{I})\mathbf{r} = 0$. We obtain for example

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ -c \end{pmatrix} \quad , \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ c \end{pmatrix}$$

Define the transformation matrix as the right eigenvector matrix

$$\mathbf{T} = [\mathbf{r}_1, \mathbf{r}_2] = \begin{pmatrix} 1 & 1 \\ -c & c \end{pmatrix}$$

The inverse of $\mathbf{T}$ is

$$\mathbf{T}^{-1} = \frac{1}{2c}\begin{pmatrix} c & -1 \\ c & 1 \end{pmatrix}$$

Therefore,

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{\Lambda}\,,$$

where

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

i.e. $\mathbf{A}$ is diagonalizable.
Summarizing, the first order system (9) is a hyperbolic system.

b) Multiplying the hyperbolic system (9) by $\mathbf{T}^{-1}$ from the left, using the diagonalization of $\mathbf{A}$ and the definition of the characteristic variables

$$\mathbf{W} = \mathbf{T}^{-1}\mathbf{U} = \frac{1}{2c}\begin{pmatrix} cB - E \\ cB + E \end{pmatrix} = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} ,$$

we get

$$\mathbf{W}_t + \mathbf{\Lambda}\mathbf{W}_x = 0 , \tag{10}$$

i.e. two scalar equations

$$\begin{aligned} w_t^1 + \lambda_1 w_x^1 &= 0 \\ w_t^2 + \lambda_2 w_x^2 &= 0 , \end{aligned}$$

which correspond to

$$\begin{aligned} \frac{dw^1}{dt} &= 0 \qquad \text{on} \qquad \tfrac{dx}{dt} = \lambda_1 \\ \frac{dw^2}{dt} &= 0 \qquad \text{on} \qquad \tfrac{dx}{dt} = \lambda_2 . \end{aligned}$$

Thus, w^1 and consequently $cB - E$ is constant on characteristics $\frac{dx}{dt} = -c$, and w^2 and consequently $cB + E$ is constant on characteristics $\frac{dx}{dt} = c$.

c) The characteristics for (9) look similar as the ones for exercise 1.2.1, Strikwerda (1989), p. 11.
If we provide the ingoing characteristic variables as boundary conditions, the hyperbolic initial boundary value problem for the hyperbolic system (9) is well-posed. For example, we can prescribe the following boundary conditions:

$$\begin{aligned} w^1(1,t) &= \alpha_1 w^2(1,t) + \beta_1(t) \\ w^2(0,t) &= \alpha_2 w^1(0,t) + \beta_2(t) \end{aligned}$$
,

where the coefficients α_1 and α_2 may be constants or functions of t. For simplicity, we assume $\alpha_1 = \alpha_2 = 0$. Then, the exact solution for the characteristic variables reads

$$\begin{aligned} w^1(x,t) &= \begin{cases} w_0^1(x+ct) & x+ct \le 1 \\ \beta_1(t + (x-1)/c) & x+ct > 1 \end{cases} \\ w^2(x,t) &= \begin{cases} w_0^2(x-ct) & x-ct \ge 0 \\ \beta_2(t - x/c) & x-ct < 0 \end{cases} \end{aligned}$$

where $w^1(x,0) = w_0^1(x)$ and $w^2(x,0) = w_0^2(x)$ are the initial conditions for the characteristic variables. Using the definition of the characteristic variables, the exact solution of (9) is obtained from

$$\mathbf{U} = \mathbf{T}\mathbf{W} = \begin{pmatrix} w^1(x,t) + w^2(x,t) \\ c(w^2(x,t) - w^1(x,t)) \end{pmatrix} . \tag{11}$$

4. a) A scheme is total variation diminishing (TVD), if

$$\sum_{j=1}^{N} |v_j^{n+1} - v_{j-1}^{n+1}| \le \sum_{j=1}^{N} |v_j^n - v_{j-1}^n| .$$

Inserting the Lax-Friedrichs method, we get using the CFL number $\sigma = a\frac{k}{h}$

$$\begin{aligned}\sum_{j=1}^{N} |v_j^{n+1} - v_{j-1}^{n+1}| &= \textstyle\sum_{j=1}^{N} |\frac{1}{2}(v_{j+1}^n - v_{j-1}^n) - \frac{\sigma}{2}(v_{j+1}^n - v_{j-1}^n) - [\frac{1}{2}(v_j^n - v_{j-2}^n) - \frac{\sigma}{2}(v_j^n - v_{j-2}^n)]| \\ &= \textstyle\sum_{j=1}^{N} |\frac{1-\sigma}{2}(v_{j+1}^n - v_j^n) + \frac{1+\sigma}{2}(v_{j-1}^n - v_{j-2}^n)| \\ &\le \textstyle\frac{1-\sigma}{2}\sum_{j=1}^{N} |v_{j+1}^n - v_j^n| + \frac{1+\sigma}{2}\sum_{j=1}^{N} |v_{j-1}^n - v_{j-2}^n| \\ &= \textstyle\sum_{j=1}^{N} |v_j^n - v_{j-1}^n|\end{aligned}$$

where the triangle inequality and $|\sigma| \le 1$ were used for the inequality and the periodic boundary conditions for the last equality.

b) The exact solution to the inviscid Burgers' equation $u_t + \left(\frac{u^2}{2}\right)_x = 0$ with the initial condition

$$u(x,0) = \begin{cases} -1 & x < x_{1+1/2} \\ 2 & x > x_{1+1/2} \end{cases} \tag{12}$$

is an expansion fan centered at $x_{1+1/2}$, i.e. $u^0_{1+1/2} = 0$. Therefore, $h^0_{1+1/2} = \frac{(u^0_{1+1/2})^2}{2} = 0$. This value agrees with $min_{-1\le u\le 2}\frac{u^2}{2} = 0$. A hyperbolic initial value problem with discontinuous initial conditions like (12) is called a Riemann problem. For the corresponding Riemann problem with

$$u(x,0) = \begin{cases} 2 & x < x_{2+1/2} \\ 1 & x > x_{2+1/2} \end{cases}$$

the exact solution is a shock moving with the shock speed $s = \frac{2+1}{2} = 1.5$. Thus, the exact solution at $x_{2+1/2}$ is $u^0_{2+1/2} = 2$. Therefore, $h^0_{2+1/2} = \frac{(u^0_{2+1/2})^2}{2} = \frac{2^2}{2} = 2$. This value agrees with $max_{2\ge u\ge 1}\frac{u^2}{2} = 2$.
The Godunov method is a conservative method

$$u_j^{n+1} = u_j^n - \frac{k}{h}(h^n_{j+1/2} - h^n_{j-1/2}) \tag{13}$$

For $j = 2$, $n = 0$ and $\frac{k}{h} = 0.25$, (13) becomes for the inviscid Burgers' equation with the numerical fluxes computed above:

$$u_2^1 = 2 - 0.25\ (2 - 0) = 1.5\ .$$

5. a) Determine the spectral radius of $\mathbf{G}^{(\mu)}$ for $\mu = 1, ..., n_0$.

$$\mathbf{G}^{(\mu)} = \begin{pmatrix} \alpha & \beta \\ \alpha & \beta\ , \end{pmatrix}$$

where $\alpha = s_\mu^2 c_\mu^{2\nu}$ and $\beta = c_\mu^2 s_\mu^{2\nu}$. The eigenvalues of $\mathbf{G}^{(\mu)}$ are easily computed to be $\lambda_1 = 0$ and $\lambda_2 = \alpha + \beta$. Thus, the spectral radius of $\mathbf{G}^{(\mu)}$ is

$$\rho(\mathbf{G}^{(\mu)} = \alpha + \beta = s_\mu^2 c_\mu^{2\nu} + c_\mu^2 s_\mu^{2\nu}\ .$$

Since $c_\mu^2 = cos^2\left(\frac{\mu\pi h_1}{2}\right) = 1 - sin^2\left(\frac{\mu\pi h_1}{2}\right) = 1 - s_\mu^2$,

$$\rho(\mathbf{G}^{(\mu)} = s_\mu^2\,(1 - s_\mu^2)^\nu + (1 - s_\mu^2)\,s_\mu^{2\nu}\ .$$

Since $0 < \mu\, h_1 \leq (n_0+1)\, h_1 = \frac{n_1+1}{2} = \frac{1}{2}$ and $0 < sin^2\left(\frac{\mu\pi h_1}{2}\right) \leq sin^2\left(\frac{1}{4}\right) = \frac{1}{2}$, we have the estimate

$$\begin{aligned} \rho(\mathbf{G}) &= max_{1\leq\mu\leq n_0+1}\{s_\mu^2\,(1-s_\mu^2)^\nu + (1-s_\mu^2)\,s_\mu^{2\nu}\} \\ &\leq max_{0\leq\xi\leq 1/2}\{\xi(1-\xi)^\nu + (1-\xi)\xi^\nu \;=:\; \rho_\nu\,. \end{aligned} \tag{14}$$

b) Since $\xi \leq 1-\xi$ for $0 \leq \xi \leq 1/2$, we can estimate the bound ρ_ν in (14) by

$$\rho(\mathbf{G}) \leq \rho_\nu \leq 2max_{0\leq\xi\leq 1/2}\{\xi(1-\xi)^\nu\}\,.$$

We define $f(\xi) = \xi(1-\xi)^\nu$ and determine the maximum $max_{0\leq\xi\leq 1/2}f(\xi)$.
For $\nu = 1$, $f'(\xi) = 1-2\xi = 0$ for $\xi = \frac{1}{2}$. It is a maximum, because $f''(\frac{1}{2}) = -2 < 0$. As $\rho(\mathbf{G}) \leq 2max_{0\leq\xi\leq 1/2}f(\xi) \leq 2f(\frac{1}{2}) = 2\frac{1}{4} = \frac{1}{2} < 1$, the TMG method is convergent for $\nu = 1$.
With similar reasoning, we get for $\nu > 1$ that $f'(\xi) = (1-\xi)^{\nu-1}\,(1-(1+\nu)\xi) = 0$ for $\xi = \frac{1}{1+\nu}$ ($\xi = 0$ is a minimum). $\xi = \frac{1}{1+\nu}$ is a maximum, because $f''(\frac{1}{1+\nu}) = (1-\xi)^{\nu-2}(-\nu) < 0$. As $\rho(\mathbf{G}) \leq 2max_{0\leq\xi\leq 1/2}f(\xi) \leq 2f(\frac{1}{1+\nu}) = 2(\frac{1}{1+\nu}(1-\frac{1}{1+\nu})^\nu) < 2\frac{1}{1+\nu} < 1$, the TMG method is also convergent for $\nu > 1$.