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Solutions to the Examination in Analysis of Numerical Methods

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1. a) Taylor expansion of the exact solution $u(x,t)$ around (x_j, t_n) yields:

$$\begin{aligned} u(x_j, t_{n+1}) &= u + ku_t + k^2/2\, u_{tt} + k^3/6\, u_{ttt} + \mathcal{O}(k^4) \\ u(x_{j\pm1}, t_n) &= u \pm hu_x + h^2/2\, u_{xx} \pm h^3/6\, u_{xxx} + \mathcal{O}(h^4) \end{aligned}$$

Inserting $u = u(x_j, t_n)$ for v_j^n into the Lax-Wendroff scheme

$$\frac{v_j^{n+1} - v_j^n}{k} + a\frac{v_{j+1}^n - v_{j-1}^n}{2h} - \left(\frac{a^2k}{2}\right)\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{h^2} = 0\,. \qquad (1)$$

and using the PDE $u_t + u_x = 0$ yields the truncation error

$$\begin{aligned} P_{k,h}u &= u_t + k/2\, u_{tt} + \mathcal{O}(k^2) + a(u_x + h^2/6\, u_{xxx} + \mathcal{O}(h^4)) - a^2k/2(u_{xx} + \mathcal{O}(h^2)) \\ &= k/2\,(u_{tt} - a^2u_{xx}) + \mathcal{O}(k^2) + \mathcal{O}(h^2) \end{aligned}$$

Using the PDE and assuming that the exact solution is smooth, we have $u_{tt} = -au_{xt} = -au_{tx} = a^2u_{xx}$. Thus, the Lax-Wendroff scheme is second order accurate in space and time, i.e. the FDM is consistent.

b) We perform the Fourier analysis by inserting the ansatz

$$v_j^n = g^n e^{i\omega x_j}$$

into the FDM (1) multiplied by k. We get after dividing by $g^{n-1}e^{i\omega x_j}$ and using Euler's formula $e^{i\Theta} = cos(\Theta) + isin(\Theta)$

$$g = 1 - i\sigma sin(\Theta) + \sigma^2(cos(\Theta) - 1)$$

where $\sigma = ak/h$ is the CFL number and $\Theta = \omega h$. Using $cos(\Theta) - 1 = -2sin^2(\Theta/2)$ and $sin(\Theta) = 2sin(\Theta/2)\sqrt{1 - sin^2(\Theta/2)}$, we obtain

$$|g|^2 = 1 - 4\sigma^2 sin^2(\Theta/2) + 4\sigma^4 sin^4(\Theta/2) + 4sin^2(\Theta/2)(1 - sin^2(\Theta/2))\,. \qquad (2)$$

Thus

$$|g|^2 = 1 - 4\sigma^2(1 - \sigma^2)sin^4(\Theta/2))\,.$$

The von Neumann condition

$$|g|^2 \le 1$$

is satisfied for $0 \le 1 - \sigma^2$, i.e. if $|\sigma| \le 1$.
Therefore, the Lax-Richtmyer equivalence theorem implies that the Lax-Wendroff scheme (1) is convergent, if $|\sigma| \le 1$.

2. We perform the GKS analysis. The stability of FDM (1) for periodic boundary conditions was shown in task 1.b).

1. Resolvent equation

Inserting the ansatz $v_j^n = z^n \tilde{v}_j$ into the FDM (1) multiplied by k yields after dividing by z^{n-1}

$$(z-1)\tilde{v}_j + \frac{\sigma}{2}\left(\tilde{v}_{j+1} - \tilde{v}_{j-1}\right) - \frac{\sigma^2}{2}\left(\tilde{v}_{j+1} - 2\tilde{v}_j + \tilde{v}_{j-1}\right) \;=\; 0 \;, \tag{3}$$

which is the resolvent equation.

2. Characteristic equation

Inserting the ansatz $\tilde{v}_j = \kappa^j$ into the resolvent equation (3), we obtain after dividing by κ^{j-1} the characteristic equation

$$(z-1)\kappa + \frac{\sigma}{2}\left(\kappa^2 - 1\right) - \frac{\sigma^2}{2}\left(\kappa^2 - 2\kappa + 1\right) \;=\; 0 \;. \tag{4}$$

Ordering the terms and assuming $0 \neq \sigma \neq 1$, (4) can also be expressed as

$$\kappa^2 + \frac{2(z-1+\sigma^2)}{\sigma - \sigma^2}\kappa - \frac{1+\sigma}{1-\sigma} \;=\; 0 \;. \tag{5}$$

3. Determinant condition

The general solution of the resolvent equation (3) is

$$\tilde{v}_j = \begin{cases} \sigma_1 \kappa_1^j + \sigma_2 \kappa_2^j & \text{if } \kappa_1 \neq \kappa_2 \\ \sigma_1 \kappa^j + \sigma_2 j \kappa^{j-1} & \text{if } \kappa_1 = \kappa_2 = \kappa \end{cases}$$

where κ_1 and κ_2 are the two roots of the characteristic equation (4). The coefficients σ_1 and σ_2 are determined such that the boundary condition $v_0^{n+1} - 2v_1^{n+1} + v_2^{n+1} = 0$ and $v^n \in l_2(0,\infty)$ are satisfied for the right quarter plane problem. (Similar reasoning is used for the left quarter plane problem with the boundary condition $v_N^{n+1} = 0$, cf. below.) Because of the form of the characteristic equation (5), the product of its roots satisfies the relation

$$\kappa_1 \kappa_2 = -\frac{1+\sigma}{1-\sigma} \;.$$

According to the hint, $|\kappa_1| < 1$ and $|\kappa_2| > 1$ for $|z| > 1$. We have to set $\sigma_2 = 0$ to secure $v^n \in l_2(0,\infty)$.
Thus, the solution must be of the form

$$v_j^n = z^n \sigma_1 \kappa_1^j \;.$$

(For the left quarter plane problem, we have to set $\sigma_1 = 0$ to secure $v^n \in l_2(0,\infty)$, since $|\kappa_1^j| \to \infty$ for $j \to \infty$ and $\kappa_1 \neq 0$. Inserting the solution into the boundary condition $v_N^{n+1} = 0$ yields $z^{n+1}\sigma_2\kappa_2^N = 0$. Thus, $\kappa_2 = 0$ for $\sigma_2 \neq 0$, and consequently $v_j^n = 0$. Therefore, the left quarter plane problem is stable.)
Inserting the solution into the boundary condition $v_0^{n+1} - 2v_1^{n+1} + v_2^{n+1} = 0$, we get

$$(1 - 2\kappa_1 + \kappa_1^2)\sigma_1 = 0 \;.$$

For $\sigma_1 \neq 0$, we obtain the case when the determinant condition is not fulfilled, i.e.

$$\kappa_1^2 - 2\kappa_1 + 1 = 0 \;. \tag{6}$$

4. Solve equations

The condition (6) has the solution $\kappa_1 = 1$, which could be a solution in the limit $\kappa_1 \to 1_-$ as $|z| \to 1_+$.
We solve the characteristic equation (4) by inserting $\kappa = 1$ into (4). We get the solution $z = 1$.
Thus, the solution is

$$\kappa_1 = 1 \quad , \, z = 1 \, . \tag{7}$$

5. Check solution

If $\kappa = 1$ and $z = 1$ were a solution in the limit $\kappa \longrightarrow 1_-$ and $z \longrightarrow 1_+$, the scheme would be unstable. To check that case, we assume that $z = 1 + \delta$ with $\delta > 0$ and $\kappa = 1 + \epsilon$. We insert z and κ into the characteristic equation (4) and check the sign of ϵ as $\delta \longrightarrow 0$. If $\epsilon < 0$, the scheme is unstable.
Inserting z and κ into (4), yields

$$\delta(1+\epsilon) + \frac{\sigma}{2}\left((1+\epsilon)^2 - 1\right) - \frac{\sigma^2}{2}\left((1+\epsilon)^2 - 2(1+\epsilon) + 1\right) \;=\; 0 \, .$$

Neglecting the second order terms, we get

$$\delta + \sigma\epsilon \;=\; 0 \, . \tag{8}$$

Thus, $\epsilon = -\delta/\sigma$. Consequently, $\epsilon > 0$, if $\sigma < 0$, and $\epsilon < 0$, if $\sigma > 0$.

6. Conclusions

Therefore, the FDM (1) with the boundary conditions stated above is unstable, if $\sigma > 0$. If $\sigma = 0$, the characteristic equation (4) becomes $z = 1$, i.e. it does not allow $z > 1$. Hence, the solution (7) cannot be a limit solution for $z \longrightarrow 1_+$. Thus, the FDM (1) with the boundary conditions stated above is stable, if $-1 \le \sigma \le 0$.

3. a) Differentiate the first and second equations of the 1D shallow water equations using the product rule:

$$\begin{aligned} h_t + uh_x + hu_x \quad &= 0 \, , \\ hu_t + u\left(h_t + (hu)_x\right) + huu_x + ghh_x \quad &= 0 \, . \end{aligned}$$

Using the first equation in the parentheses of the second one and dividing the second equation by h assuming that the water depth h is positive, we obtain the nonlinear system

$$\mathbf{V}_t + \mathbf{B}\mathbf{V}_x = 0 \, , \tag{9}$$

where

$$\mathbf{V} = \begin{pmatrix} h \\ u \end{pmatrix} , \quad \mathbf{B} = \begin{pmatrix} u & h \\ g & u \end{pmatrix} .$$

To linearize (9) around the constant reference state $\mathbf{V}_0 = [H, U]^T$, we insert the ansatz $\mathbf{V} = \mathbf{V}_0 + \mathbf{U}$ into (9), where $\mathbf{U} = [b, v]^T$ denotes the vector of the water depth and velocity perturbations. Since the time and space derivatives of $\mathbf{V}_0$ are zero, we obtain

$$\begin{aligned} b_t + Ub_x + vb_x + Hv_x + bv_x \quad &= 0 \, , \\ v_t + gb_x + Uv_x + vv_x \quad &= 0 \, . \end{aligned}$$

As the perturbations and their derivatives are assumed to be small compared to their reference states, the nonlinear terms vb_x, bv_x and vv_x in the above equations are neglected. We obtain the linearized 1D shallow water equations

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = 0 \ , \tag{10}$$

where

$$\mathbf{U} = \begin{pmatrix} b \\ v \end{pmatrix} \ , \quad \mathbf{A} = \begin{pmatrix} U & H \\ g & U \end{pmatrix} \ .$$

b) Determine eigenvalues λ of $\mathbf{A}$:

$$det(\mathbf{A} - \lambda \mathbf{I}) = (U - \lambda)^2 - c^2 = 0 \Longleftrightarrow \lambda_{1/2} = U \mp c \ ,$$

where $c = \sqrt{gH}$ is the gravity wave speed. Thus, the eigenvalues $\lambda_1 = U - c$ and $\lambda_2 = U + c$ are real. As they are distinct, the corresponding eigenvectors are linearly independent. Thus, the system (10) is hyperbolic.
Assume that the eigenvectors $\mathbf{r}_1$ and $\mathbf{r}_2$ define the transformation matrix $\mathbf{T} = [\mathbf{r}_1, \mathbf{r}_2]$. As $\mathbf{AT} = \mathbf{T\Lambda}$, where

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \ ,$$

the system (10) can be diagonalized bt multiplying it with $\mathbf{T}^{-1}$ from the left. Defining the characteristic variables $\mathbf{W} = \mathbf{T}^{-1}\mathbf{U}$, we obtain

$$\mathbf{W}_t + \mathbf{\Lambda}\mathbf{W}_x = 0 \ , \tag{11}$$

i.e. two scalar equations

$$\begin{aligned} w_t^1 + \lambda_1 w_x^1 &= 0 \\ w_t^2 + \lambda_2 w_x^2 &= 0 \ , \end{aligned}$$

which correspond to

$$\begin{aligned} \frac{dw^1}{dt} &= 0 \qquad \text{on} \qquad \tfrac{dx}{dt} = \lambda_1 \\ \frac{dw^2}{dt} &= 0 \qquad \text{on} \qquad \tfrac{dx}{dt} = \lambda_2 \ . \end{aligned}$$

Thus, the first characteristic variable w^1 is constant on characteristics $\frac{dx}{dt} = U - c$, and the second characteristic variable w^2 is constant on characteristics $\frac{dx}{dt} = U + c$.

c) The gravity wave speed $c = \sqrt{gH}$ here is $c = 100\frac{m}{s}$. Thus, the slopes of the left and right going characteristics are $\frac{dx}{dt} = \lambda_1 = -100\frac{m}{s}$ and $\frac{dx}{dt} = \lambda_2 = 100\frac{m}{s}$, respectively, cf. Fig. 1. The exact solution of the problem corresponds to the quiescent initial condition $b = 0m$ and $v = 0\frac{m}{s}$, except for two strips in the x-t diagram. In the strip (L) bounded by the left going characteristcs with slope $\frac{dx}{dt} = -100\frac{m}{s}$ starting at $x = -10m$ and $x = 10m$, w^1 is constant and equal to the initial condition of w^1 in the interval $(-10m, 10m)$. In the strip (R) bounded by the right going characteristcs with slope $\frac{dx}{dt} = 100\frac{m}{s}$ starting at $x = -10m$ and $x = 10m$, w^2 is constant and equal to the initial condition of w^2 in the interval $(-10m, 10m)$.
To construct the exact solution, we determine the eigenvectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of $\mathbf{A}$ (with $U = 0\frac{m}{s}$) corresponding to the eigenvalues $\lambda_1 = -c$ and $\lambda_2 = c$. We obtain

$$\mathbf{r}_1 = \begin{pmatrix} \sqrt{H} \\ -\sqrt{g} \end{pmatrix} \ , \quad \mathbf{r}_2 = \begin{pmatrix} \sqrt{H} \\ \sqrt{g} \end{pmatrix}$$

Define the transformation matrix as the right eigenvector matrix

$$\mathbf{T} = [\mathbf{r}_1, \mathbf{r}_2] = \begin{pmatrix} \sqrt{H} & \sqrt{H} \\ -\sqrt{g} & \sqrt{g} \end{pmatrix}$$

The inverse of $\mathbf{T}$ is

$$\mathbf{T}^{-1} = \frac{1}{2c} \begin{pmatrix} \sqrt{g} & -\sqrt{H} \\ \sqrt{g} & \sqrt{H} \end{pmatrix}$$

The characteristic variables become

$$\mathbf{W} = \mathbf{T}^{-1}\mathbf{U} = \frac{1}{2c} \begin{pmatrix} \sqrt{g}b - \sqrt{H}v \\ \sqrt{g}b + \sqrt{H}v \end{pmatrix} = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} .$$

In the intersection (I) of strips (L) and (R), we get as $b = 10m$ and $v = 0\frac{m}{s}$ (omitting the dimensions)

$$\mathbf{W}_I = \frac{1}{2c} \begin{pmatrix} \sqrt{g}10 \\ \sqrt{g}10 \end{pmatrix} . \qquad (12)$$

In the left strip (L), except for the intersection (I), we get as $w^2 = 0$ there

$$\mathbf{W}_L = \frac{1}{2c} \begin{pmatrix} \sqrt{g}10 \\ 0 \end{pmatrix} . \qquad (13)$$

In the right strip (R), except for the intersection (I), we get as $w^1 = 0$ there

$$\mathbf{W}_R = \frac{1}{2c} \begin{pmatrix} 0 \\ \sqrt{g}10 \end{pmatrix} . \qquad (14)$$

We get the solution from

$$\mathbf{U} = \mathbf{T}\mathbf{W} = \begin{pmatrix} \sqrt{H}w^1 + \sqrt{H}w^2 \\ -\sqrt{g}w^1 + \sqrt{g}w^2 \end{pmatrix} .$$

Thus, in the intersection (I) of strips (L) and (R), we obtain with $c = \sqrt{gH}$

$$\mathbf{U}_I = \frac{1}{2c} \begin{pmatrix} \sqrt{gH}10 + \sqrt{gH}10 \\ -g10 + g10 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} . \qquad (15)$$

In the left strip (L), except for the intersection (I), we get with $H = 1000m$ and $g = 10m/s^2$

$$\mathbf{U}_L = \frac{1}{2c} \begin{pmatrix} \sqrt{gH}10 \\ -g10 \end{pmatrix} = \begin{pmatrix} 5 \\ -\sqrt{g/H} \end{pmatrix} = \begin{pmatrix} 5 \\ -0.5 \end{pmatrix} . \qquad (16)$$

In the right strip (R), except for the intersection (I), we get

$$\mathbf{U}_R = \frac{1}{2c} \begin{pmatrix} \sqrt{gH}10 \\ g10 \end{pmatrix} = \begin{pmatrix} 5 \\ \sqrt{g/H} \end{pmatrix} = \begin{pmatrix} 5 \\ 0.5 \end{pmatrix} . \qquad (17)$$

Thus, the water height above the water level is $10m$ and the velocity zero in the intersection (I) near the origin of the tsunami, whereas the water level is $5m$ and the velocity $-0.5\frac{m}{s}$ and $0.5\frac{m}{s}$ in the left and right strips, respectively.

The left and right going gravity waves, i.e. the fronts of the left and right strips, travel with the speed $-c = -100\frac{m}{s}$ to the left and with the speed $c = 100\frac{m}{s}$ to the right, respecticely. Thus, the time needed for a front of the tsunami to reach a coast $1000km$ away from its origin is

$$T = \frac{10^6 m}{100\frac{m}{s}} = 10^4 s = 2h\ 46min\ 40s .$$

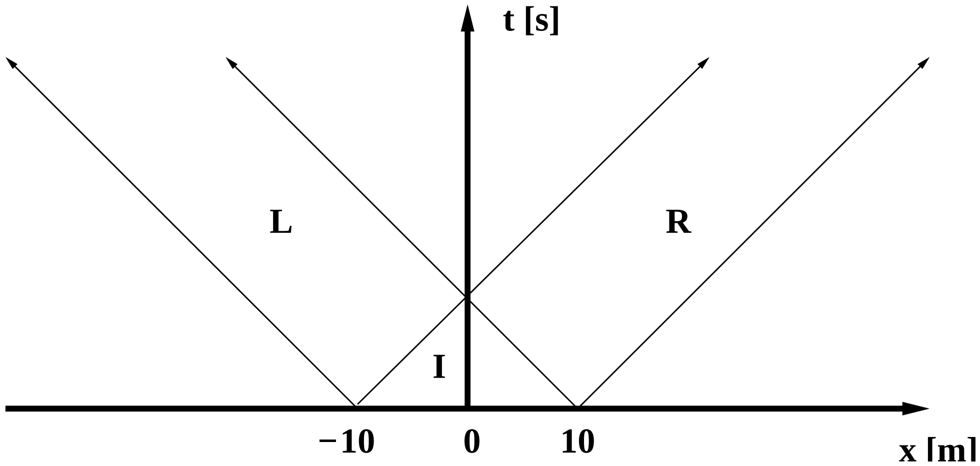

Figure 1: Characteristics of discontinuities starting at $-10m$ and $10m$.

4. a) A scheme is total variation diminishing (TVD), if

$$TV(\mathbf{v}^{n+1}) \leq TV(\mathbf{v}^n) \ ,$$

where $TV(\mathbf{v}^n) = \sum_{j=1}^{N} |v_j^n - v_{j-1}^n|$ is the total variation of $\mathbf{v}^n$.
Suppose the initial condition is defined by a discontinuity at the midpoint $x_{i+1/2} = (x_i + x_{i+1})/2$ between the grid points x_i and x_{i+1}:

$$u(x,0) = \begin{cases} u_L & x < x_{i+1/2} \\ u_R & x > x_{i+1/2} \end{cases}$$

with $u_L \neq u_R$. We apply the Lax-Wendroff method

$$v_j^{n+1} = v_j^n - \frac{\sigma}{2}(v_{j+1}^n - v_{j-1}^n) + \frac{\sigma^2}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) \ , \tag{18}$$

where $\sigma = a\frac{k}{h}$ is the CFL number, to compute

$$\begin{aligned} v_j^1 &= \begin{cases} u_L & j \leq i-1 \\ u_R & j \geq i+2 \end{cases} \\ v_i^1 &= u_L + \tfrac{u_L - u_R}{2}\sigma(1-\sigma) \\ v_{i+1}^1 &= u_R + \tfrac{u_L - u_R}{2}\sigma(1+\sigma) \end{aligned}$$

We get

$$\begin{aligned} TV(\mathbf{v}^1) &= |v_i^1 - v_{i-1}^1| + |v_{i+1}^1 - v_i^1| + |v_{i+2}^1 - v_{i+1}^1| \\ &= |\frac{u_L - u_R}{2}\sigma(1-\sigma)| + |(u_L - u_R)(\sigma^2 - 1)| + |\frac{u_L - u_R}{2}\sigma(1+\sigma)| \end{aligned}$$

Thus, we get

$$TV(\mathbf{v}^1) = \begin{cases} |u_L - u_R|(1 + \sigma - \sigma^2) & 0 < \sigma < 1 \\ |u_L - u_R|(1 - \sigma - \sigma^2) & -1 < \sigma < 0 \end{cases}$$

Therefore,

$$TV(\mathbf{v}^1) > |u_L - u_R| = TV(\mathbf{v}^0) \, .$$

That means that we have proved with a counterexample that the Lax-Wendroff scheme is not TVD.

b) Using the definition of the energy, differentiating and replacing $(v_j)_t$ by $-\frac{1}{3}\left(v_j D_0 v_j + D_0(v_j^2)\right)$, we obtain

$$\begin{aligned}
\frac{d}{dt}||v||_h^2 &= \frac{d}{dt}h\sum_{j=1}^{N} v_j^2 = h\sum_{j=1}^{N} 2v_j(v_j)_t \\
&= -\frac{2h}{3}\sum_{j=1}^{N} v_j\left(v_j D_0 v_j + D_0(v_j^2)\right) \\
&= -\frac{1}{3}\sum_{j=1}^{N}\left(v_j^2(v_{j+1}-v_{j-1}) + v_j(v_{j+1}^2 - v_{j-1}^2)\right) \\
&= -\frac{1}{3}\left(\sum_{j=1}^{N} v_j^2 v_{j+1} - \sum_{j=0}^{N-1} v_{j+1}^2 v_j + \sum_{j=1}^{N} v_j v_{j+1}^2 - \sum_{j=0}^{N-1} v_{j+1} v_j^2\right) \\
&= -\frac{1}{3}\left((\sum_{j=1}^{N} v_j^2 v_{j+1} - \sum_{j=0}^{N-1} v_{j+1} v_j^2) + (-\sum_{j=0}^{N-1} v_{j+1}^2 v_j + \sum_{j=1}^{N} v_j v_{j+1}^2)\right) \\
&= -\frac{1}{3}\left((v_N^2 v_{N+1} - v_1 v_0^2)) + (-v_1^2 v_0 + v_N v_{N+1}^2)\right) \\
&= 0\,,
\end{aligned}$$

as the last expression is zero because of the periodic boundary conditions. Thus, we have

$$\frac{d}{dt}||v||_h^2 = 0\,.$$

Integrating over time yields $||v(t)||_h^2 - ||v(0)||_h^2 = 0$. Thus, we obtain the estimate for the 2-norm

$$||v(t)||_h \le ||v(0)||_h\,.$$

As the 2-norm of the semidiscrete solution v at time t is bounded by the 2-norm of the initial condition, the proposed semidiscretization of the inviscid Burgers' equation is stable.

5. a) Suppose $\mathbf{w}_\mu$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $\zeta_\mu = \frac{4}{h^2}\sin^2(\frac{\mu\pi h}{2})$, i.e. $\mathbf{A}\mathbf{w}_\mu = \zeta_\mu \mathbf{w}_\mu$, $\mu = 1, \dots, n$. Since $\mathbf{D}^{-1} = \frac{h^2}{2}\mathbf{I}$, we obtain for the iteration matrix

$$\mathbf{G} = \mathbf{I} - \frac{1}{2}\mathbf{D}^{-1}\mathbf{A} = \mathbf{I} - \frac{h^2}{4}\mathbf{A}\,.$$

Thus, we obtain

$$\mathbf{G}\mathbf{w}_\mu = (\mathbf{I} - \frac{h^2}{4}\mathbf{A})\mathbf{w}_\mu = (1 - \frac{h^2}{4}\zeta_\mu)\mathbf{w}_\mu\,.$$

Consequently, the eigenvalues of $\mathbf{G}$ are

$$\lambda_\mu = 1 - \frac{h^2}{4}\zeta_\mu = 1 - \sin^2(\frac{\mu\pi h}{2}) = \cos^2(\frac{\mu\pi h}{2})\,, \mu = 1, \dots, n. \tag{19}$$

For $n_1 = n$ odd, $h_1 = h$, $n_0 = \frac{n_1 - 1}{2}$, $h_0 = 2h_1$, $\mu' = n_1 + 1 - \mu$, we get

$$\lambda_{\mu'} = \cos^2(\frac{\mu'\pi h_1}{2}) = \cos^2(\frac{(n_1+1-\mu)\pi h_1}{2}) = \sin^2(\frac{\mu\pi h_1}{2})\,,$$

where we used $(n_1+1)h_1 = 1$ and the addition formula for cos. As $(n_0+1)h_1 = 1/2$ and $\cos^2(\frac{(n_0+1)\pi h_1}{2}) = \sin^2(\frac{(n_0+1)\pi h_1}{2}) = 1/2$, the error modes $\mathbf{w}_\mu$ with low wave numbers $\mu = 1, ..., n_0$ are damped with a factor larger than $1/2$, whereas the error modes $\mathbf{w}_\mu$ with high wave numbers $\mu' = n_1 + 1 - \mu$ are damped with a factor lower than $1/2$. Very high high wave number error modes are damped very quickly, because in each damped Jacobi iteration those error modes are reduced by a factor of $\lambda_{\mu'} \approx 0$.

Error modes $\mathbf{w}_\mu$ with low wave numbers μ on the fine grid, where $1/4 \leq \mu h_1 < 1/2$, become high wave numbers on the coarse grid, because the wave number μ is seen on the coarse grid with $h_0 = 2h_1$ as $1/2 \leq \mu h_0 < 1$ and the error mode $\mathbf{w}_\mu$ is damped by the factor $\lambda_\mu = \cos^2(\frac{\mu\pi h_0}{2}) \leq 1/2$ instead of $\lambda_\mu = \cos^2(\frac{\mu\pi h_1}{2}) > 1/2$ on the fine grid. In the same way, error modes with wave numbers μ on the coarse grid, where $1/4 \leq \mu h_0 < 1/2$, become high wave numbers on the next coarser grid and can be quickly damped there, etc.

The combination of residual smoothing on the fine grid, where the high wave number error modes are efficiently damped by the damped Jacobi method in our case, and the correction on the coarse grid, where the low wave numbers on the fine grid become high wave numbers and are efficiently damped, leads to the efficiency of the multigrid method.

b) The error $\mathbf{e}^{(m)}$ at iteration m is related to the error $\mathbf{e}^{(0)}$ of the starting guess by

$$\mathbf{e}^{(m)} = \mathbf{G}^m \mathbf{e}^{(0)}\,,$$

where $\mathbf{G}$ is the iteration matrix. In the 2-norm (Euclidian norm here), we have the estimate

$$||\mathbf{e}^{(m)}||_2 \leq ||\mathbf{G}||_2^m ||\mathbf{e}^{(0)}||_2\,.$$

If the error is to be reduced by one decimal digit, we require $||\mathbf{e}^{(m)}||_2 \leq 10^{-1}||\mathbf{e}^{(0)}||_2$. Requiring $||\mathbf{G}||_2^m ||\mathbf{e}^{(0)}||_2 \leq 10^{-1}||\mathbf{e}^{(0)}||_2$, we get the sufficient number of iterations m to reduce the error by one decimal digit:

$$m \geq \frac{-1}{log_{10}(||\mathbf{G}||_2)}\,. \tag{20}$$

Only if $\mathbf{G}$ is symmetric, $\rho(\mathbf{G}) = ||\mathbf{G}||_2$ is guaranteed. Otherwise $\frac{-1}{log_{10}(\rho(\mathbf{G}))}$ gives only an approximation of the number of iterations to reduce the error by one decimal digit.

For the multigrid method, we may assume that $||\mathbf{G}||_2 = 0.1$. Then, (20) implies that one iteration is sufficient to reduce the error by one decimal digit.