Uppsala University
Department of Information Technology
Division of Scientific Computing

Examination: Analysis of Numerical Methods, 2001-10-12

Time: $15^{00} - 20^{00}$
Tools: Mathematical Handbook
Each task can give 6 points. To get full points you are required to show your computations in detail and motivate your reasoning.

1. Given the differential equation
$$u_t - \lambda u_{xx} = f(x) \quad \lambda > 0$$
on the interval $x \in [\,0\,,1\,]$ with periodic boundary conditions, we apply the numerical scheme
$$\left(1 + \frac{h^2}{12} D_+ D_-\right) \cdot \left(\frac{u_j^{n+1} - u_j^n}{k}\right) - \lambda D_+ D_- u_j^n = \left(1 + \frac{h^2}{12} D_+ D_-\right) f_j$$
Show that the scheme is *convergent* for $\frac{\lambda k}{h^2} \leq \frac{1}{3}$. (Hint: Show that $D_+ D_- u = (1 + \frac{h^2}{12} D_+ D_-) \frac{\partial^2 u}{\partial x^2} + O(h^4)$)

2. We want to solve the problem
$$\begin{aligned} &u_t - a(x) \cdot u_x = f(x,t), \quad 0 < x < 1 \\ &u(x,0) = h(x) \\ &u(1,t) = g(t) \end{aligned}$$
where $a(x) = \pi^2 \cos(\frac{\pi x}{2})$. We apply the leapfrog scheme with the numerical boundary condition
$$v_0^{n+1}(1 + a_{1/2}\lambda) = a_{1/2}\lambda v_1^{n+1} + v_0^n$$
where $a_{1/2} = (a(x_0) + a(x_1))/2$ and $\lambda = dt/dx$.
Investigate for which λ the numerical method is stable. Use the GKSO analysis.

3. $D_+ D_-(I - \frac{h^2}{12} D_+ D_-) v_j$ approximates $d^2 v / dx^2$ with 4th order accuracy. We want to use that operator to get a fourth order approximation to the problem
$$\begin{aligned} &u_{xx} + u_{yy} = f, \;\; 0 < x < 1, \; 0 < y < 1 \\ &u = g(x,y) \;\; \text{at the boundary}, \quad g(x,y) \in C^\infty \end{aligned}$$
In which points are extra numerical boundary conditions required? Derive 4th order boundary conditions for those points.

4. Under certain circumstances, the stability theory for linear problems can be applied to nonlinear problems. Describe when and how. You may treat the scalar model problem $u_t + f(u)_x = 0$, where $f(u)$ is nonlinear in u.

   Moreover, nonlinear problems require special properties (besides consistency and stability) for the numerical scheme in order that those yield a good solution. Give examples of three such properties and explain their significance.

5. Consider the model problem
$$\begin{aligned} -\phi'' &= \pi^2 \sin \pi x, \qquad 0 < x < 1, \\ \phi(0) &= \phi(1) = 0 \end{aligned}$$

If the equation is discretized by D_+D_-, we get the linear system of equations $Au = f$, where

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \qquad (\, n \times n \,).$$

We want to solve the system with an l-level *multigrid method*. One of the ideas behind multigrid is to first damp the high frequencies in the error with a simple iterative method (e.g. Jacobi) and then transfer the solution to a coarser grid. If the high frequencies are not completely damped, what happens with them at the restriction? How do they get in to the solution on the coarse grid? Do a mathematical investigation. Assume that we apply the restriction operator with direct injection as restriction, and compare the results. For your help, you have that the eigenvectors of A are given by

$$v_\mu = \sqrt{2h}\begin{bmatrix} \sin(\mu\pi 1h) \\ \sin(\mu\pi 2h) \\ \vdots \\ \sin(\mu\pi nh) \end{bmatrix}$$

with corresponding eigenvalues $\lambda_\mu = \frac{4}{h^2}\sin^2(\frac{\mu\pi h}{2})$, $\mu = 1, \dots, n$.

Good luck!