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Examination in Analysis of Numerical Methods

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Time: 9.00-14.00 h.
Tools: Beta Mathematics Handbook.
Maximum number of points is 30. To get full points you must show your computations in detail and motivate your assumptions.

1. Consider the PDE problem

$$\begin{cases} u_t + au_x = bu & x \in [0,1],\ t \geq 0, \\ u(x,0) = f(x), & x \in [0,1], \\ u(0,t) = u(1,t), & t \geq 0, \end{cases} \tag{1}$$

where $a < 0$ and $b \leq 0$ are constants. We want to solve (1) with the implicit finite difference method

$$\frac{v_j^{n+1} - v_j^n}{k} + a\frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2h} = bv_j^{n+1} . \tag{2}$$

   a) Prove that the scheme (2) is convergent.

   b) Suppose that we replace the periodic boundary conditions by non-periodic ones. How would you choose the time spacing k to obtain the fastest convergence to the steady state? How many time steps would you need?

(6p)

2. Consider the implicit finite difference method (2) for $b \equiv 0$ with the boundary conditions

$$\begin{cases} v_0^{n+1} = 2v_1^{n+1} - v_2^{n+1}, \\ v_N^{n+1} = 0. \end{cases} \tag{3}$$

Check the stability of scheme (2) with the boundary conditions (3). You may use results from task 1. here. (6p)

3. The acoustic pressure p and the acoustic velocity v in a one-dimensional fluid flow with velocity u and speed of sound c are governed by the first order system

$$\begin{pmatrix} v \\ p \end{pmatrix}_t + \begin{pmatrix} u & c \\ c & u \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix}_x = 0 , \tag{4}$$

where u and $0 < c$ are assumed to be constant.

   a) Show that (4) is a hyperbolic system, i.e. that the coefficient matrix $\mathbf{A}$ in (4) has real eigenvalues and is diagonalizable.

b) What are the slopes of the characteristics and which quantities are constant along them?

c) State the leapfrog scheme to solve the system (4) numerically. Derive the stability condition, which has to be respected for periodic boundary conditions.

(6p)

4. a) Show that the upwind method

$$v_j^{n+1} = v_j^n - a\lambda(v_j^n - v_{j-1}^n) \tag{5}$$

for the one-way wave equation $u_t + au_x = 0$, $0 < a$ constant, with periodic boundary conditions is stable and total variation diminishing for $a\lambda \leq 1$.

b) Devise a conservative upwind method for the inviscid Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{6}$$

and state the numerical flux function.
Apply the method to compute one time step with the initial condition

$$u(x,0) = \begin{cases} -1 & \text{if } x < 0 \\ +1 & \text{if } x > 0 \end{cases} \tag{7}$$

Assume grid points $x_{j+1/2} = (j+\frac{1}{2})h$ with integers j. What kind of problem occurs? Suggest a solution.

(6p)

5. a) Consider the linear system $\mathbf{Au} = \mathbf{f}$ arising from discretizing the ODE

$$\begin{cases} -u'' = f(x)\,, & 0 \leq x \leq 1 \\ u(0) = u(1) = 0 \end{cases} \tag{8}$$

by $-D_+D_-u_j = f_j$, $j = 1, ..., n$, with $h = \frac{1}{n+1}$.
Discuss how two basic ideas are used to devise the two-grid multigrid algorithm for solving $\mathbf{Au} = \mathbf{f}$. State the algorithm and derive its iteration matrix $\mathbf{G}$.

b) Consider the Dirichlet problem for the Poisson equation on the unit square $\Omega = [0,1] \times [0,1]$, i.e.

$$\begin{cases} -(u_{xx} + u_{yy}) = f(x,y)\,, & (x,y)^T \in \Omega \\ u(x,y) = 0\,, & (x,y)^T \in \partial\Omega \end{cases} \tag{9}$$

Assume that the Poisson equation is discretized by the central second order accurate finite difference approximation on a uniform grid with the grid points (x_i, y_j), where $x_i = ih$, $i = 0, 1, ..., n, n+1$, $y_j = jh$, $j = 0, 1, ..., n, n+1$, and the grid spacing $h = \frac{1}{n+1}$.
With a rowwise ordering from the bottom to the top, i.e.
$\mathbf{U} = [u_{1,1}, u_{2,1}, ..., u_{n,1}, u_{1,2}, u_{2,2}, u_{n,2}, ..., u_{1,n}, u_{2,n}, ..., u_{n,n}]^T$,
we get the linear system $\mathbf{Au} = \mathbf{f}$. State the structure of the matrix $\mathbf{A}$. Compare the operation count in terms of $O(n)$ floating point operations for solving the linear system with a direct method, standard iterative methods and the multigrid method. In your reasoning, you may assume that the convergence results for solving $\mathbf{Au} = \mathbf{f}$ for the elliptic PDE problem (9) and the ODE boundary value problem (8) are similar.

(6p)

Good luck!