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Examination in Analysis of Numerical Methods

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Time: 9.00-14.00 h.
Tools: Beta Mathematics Handbook.
Maximum number of points is 30. To get full points you must show your computations in detail and motivate your assumptions.

1. Consider the PDE problem

$$\begin{cases} u_t = bu_{xx} & x \in [0,1],\ t \geq 0, \\ u(x,0) = f(x), & x \in [0,1], \\ u(0,t) = u(1,t), & t \geq 0, \end{cases} \tag{1}$$

where $b > 0$ is constant. We want to solve (1) with the Du Fort-Frankel scheme

$$\frac{v_j^{n+1} - v_j^{n-1}}{2k} = b\frac{v_{j+1}^n - v_j^{n+1} - v_j^{n-1} + v_{j-1}^n}{h^2}\,. \tag{2}$$

a) Show that the Du Fort-Frankel scheme (2) is second order accurate in space and time, if $\mu = \frac{k}{h^2}$ is constant. With which PDE is (2) consistent, if $\lambda = \frac{k}{h}$ is constant?

b) Prove that the Du Fort-Frankel scheme (2) is unconditionally stable.

(6p)

2. Consider the Du Fort-Frankel scheme (2) with the boundary conditions

$$\begin{cases} \frac{-3v_0^{n+1} + 4v_1^{n+1} - v_2^{n+1}}{2h} = 0, \\ v_N^{n+1} = 0 \end{cases} \tag{3}$$

discretizing homogeneous Neumann and Dirichlet boundary conditions, respectively. Check the stability of scheme (2) with the boundary conditions (3). You may use results from task 1.
Hint: If the asymptotic analysis for $\delta \to 0$ fails to give an answer with the first order terms, the second order terms have to be checked. $\epsilon\, f(\delta, \epsilon) = \delta^2$ implies $f(\delta, \epsilon) \to 0$ for $\delta \to 0$. (6p)

3. Electromagnetic phenomena are governed by the Maxwell equations. We consider the 1D Maxwell equations in a nonconducting medium:

$$B_t + E_x = 0 \tag{4}$$

$$E_t + c^2 B_x = 0 \tag{5}$$

where B is the magnetic induction and E the electric field. c is the propagation speed of the electromagnetic wave, i.e. c is a positive constant.

a) Show that the 1D Maxwell equations (4) and (5) form a hyperbolic system, i.e. that the coefficient matrix $\mathbf{A}$ defined by (4) and (5) has real eigenvalues and is diagonalizable.

b) What are the slopes of the characteristics and which quantities are constant along them?

c) Suppose we want to solve the 1D Maxwell equations (4) and (5) on the interval $[0, 1]$. Provide boundary conditions such that the initial boundary value problem for the 1D Maxwell equations is well-posed. Sketch the characteristics and construct the exact solution.

(6p)

4. a) Show that the Lax-Friedrichs scheme

$$v_j^{n+1} = \frac{1}{2}(v_{j+1}^n + v_{j-1}^n) - akD_0v_j^n \tag{6}$$

for the one-way wave equation $u_t + au_x = 0$, a constant, with periodic boundary conditions is total variation diminishing, if $|a|\frac{k}{h} \leq 1$.

b) The Godunov method is a conservative method for scalar conservation laws

$$u_t + f(u)_x = 0\,. \tag{7}$$

The numerical flux function of the Godunov method is computed by

$$h_{j+1/2}^n = f(u_{j+1/2}^n)\,, \tag{8}$$

where $u_{j+1/2}^n$ is the exact solution of (7) at $x_{j+1/2} = \frac{1}{2}(x_j + x_{j+1})$ and $t > 0$ for the initial condition

$$u(x,0) = \begin{cases} u_j^n & x < x_{j+1/2} \\ u_{j+1}^n & x > x_{j+1/2} \end{cases} \tag{9}$$

Check that the formula

$$f(u_{j+1/2}^n) = \begin{cases} min_{u_j^n \leq u \leq u_{j+1}^n} f(u) & u_j^n \leq u_{j+1}^n \\ max_{u_j^n \geq u \geq u_{j+1}^n} f(u) & u_j^n > u_{j+1}^n \end{cases} \tag{10}$$

for convex fluxes, i.e. $\frac{d^2f(u)}{du^2} > 0$, is correct for the flux function $f(u) = \frac{u^2}{2}$ of the inviscid Burgers' equation with $[u_1^0, u_2^0, u_3^0] = [-1, 2, 1]$. Thus, compute $h_{1+1/2}^0$ and $h_{2+1/2}^0$ by solving (7) and (9) for the given data and compare with (10).
Compute u_2^1 with the Godunov method and $\frac{k}{h} = 0.25$.

(6p)

5. Consider the linear system $\mathbf{Au} = \mathbf{f}$ arising from discretizing the ODE

$$\begin{cases} -u'' = f(x)\,, & 0 \leq x \leq 1 \\ u(0) = u(1) = 0 \end{cases} \tag{11}$$

by $-D_+D_-u_j = f_j$, $j = 1, ..., n_1$, with $h_1 = \frac{1}{n_1+1}$.

a) Estimate the spectral radius of the iteration matrix $\mathbf{G}$ for the two-grid multigrid method by using (proof not required)

$$\rho(\mathbf{G}) = max_{1\leq\mu\leq n_0+1}\{\rho(\mathbf{G}^{(\mu)})\}\,, \tag{12}$$

where

$$\mathbf{G}^{(\mu)} = \begin{pmatrix} s_\mu^2 & c_\mu^2 \\ s_\mu^2 & c_\mu^2 \end{pmatrix} \begin{pmatrix} c_\mu^{2\nu} & 0 \\ 0 & s_\mu^{2\nu} \end{pmatrix}, \quad \mu = 1, ..., n_0, \tag{13}$$

and

$$\mathbf{G}^{(n_0+1)} = 2^{-\nu}$$

with $n_0 + 1 = (n_1 + 1)/2$, $s_\mu^2 = sin^2\left(\frac{\mu\pi h_1}{2}\right)$ and $c_\mu^2 = cos^2\left(\frac{\mu\pi h_1}{2}\right)$.
The upper bound of $\rho(\mathbf{G})$ should be as sharp as possible.

b) Using the result from (a), prove that the two-grid multigrid method is convergent for $\nu = 1, 2, ...,$ where ν is the number of damped Jacobi iterations.
Hint: If you have not solved (a), you may use the estimate

$$\rho(\mathbf{G}(\nu)) \leq 2max_{0\leq\xi\leq 1/2}\{\xi(1-\xi)^\nu\}\,. \tag{14}$$

(6p)

Good luck!