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Examination in Analysis of Numerical Methods

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Time: 9.00-14.00 h.
Tools: Beta Mathematics Handbook.
Maximum number of points is 30. To get full points you must show your computations in detail and motivate your assumptions.

1. Consider the PDE problem

$$\begin{cases} u_t + a u_x = 0 & x \in [0,1],\, t \geq 0, \\ u(x,0) = f(x), & x \in [0,1], \\ u(0,t) = u(1,t), & t \geq 0, \end{cases} \tag{1}$$

where a is constant. We want to solve (1) with the Lax-Wendroff scheme

$$\frac{v_j^{n+1} - v_j^n}{k} + a\frac{v_{j+1}^n - v_{j-1}^n}{2h} - \left(\frac{a^2 k}{2}\right)\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{h^2} \;=\; 0\,. \tag{2}$$

a) Show that the Lax-Wendroff scheme (2) is second order accurate in space and time.

b) Prove that the Lax-Wendroff scheme (2) is stable, if the CFL condition $|\sigma| = |a\frac{k}{h}| \leq 1$ is satisfied. When is the Lax-Wendroff scheme (2) convergent?

(6p)

2. Consider the Lax-Wendroff scheme (2) with the boundary condition

$$\begin{cases} v_0^{n+1} = 2v_1^{n+1} - v_2^{n+1}, \\ v_N^{n+1} = 0 \end{cases} \tag{3}$$

discretizing homogeneous Neumann and Dirichlet boundary conditions, respectively. When is the scheme (2) with the boundary conditions (3) stable? You may use results from task 1.
Hint: For the roots κ_1 and κ_2 of the characteristic equation, you may assume that $|\kappa_1| < 1$ and $|\kappa_2| > 1$ for $|z| > 1$. (6p)

3. Waves in shallow water are governed by the shallow water equations. We consider the 1D shallow water equations for a flat bottom:

$$\begin{pmatrix} h \\ hu \end{pmatrix}_t + \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}_x \;=\; 0\,, \tag{4}$$

where h here is the water depth, u the fluid velocity, and $g \approx 10\frac{m}{s^2}$ the gravitational constant.

a) The 1D shallow water equations (4) are written as a conservation law. Express the 1D shallow water equations (4) as a nonlinear system $\mathbf{V}_t + \mathbf{B}(\mathbf{V})\,\mathbf{V}_x = 0$ with $\mathbf{V} = [h, u]^T$. Linearize that nonlinear system around a constant water depth H and a constant velocity U.

b) Show that the linearized 1D shallow water equations

$$\begin{pmatrix} b \\ v \end{pmatrix}_t + \begin{pmatrix} U & H \\ g & U \end{pmatrix} \begin{pmatrix} b \\ v \end{pmatrix}_x = 0\,, \tag{5}$$

where b and v denote water depth and velocity perturbations, respectively, form a hyperbolic system, i.e. that the coefficient matrix defined by (5) has real eigenvalues and is diagonalizable. What are the slopes of the characteristics? Motivate which quantities are constant along them?

c) Suppose we want to solve the linearized 1D shallow water equations (5) for $H = 1000\ m$ and $U = 0\ \frac{m}{s}$ with the initial conditions $v(x, 0) = 0\ \frac{m}{s}$ and

$$b(x,0) = \begin{cases} 10m & \text{if } |x| < 10\,m \\ 0m & \text{if } |x| > 10\,m \end{cases} \tag{6}$$

modeling the initial conditions of a tsunami. v describes the fluid velocity in initially quiescent water and b the water height above the water surface. Sketch the characteristics and construct the exact solution. When will the tsunami reach a coast, which is 1000 km away from its origin?

(6p)

4. a) Show that the Lax-Wendroff scheme (2) for the one-way wave equation $u_t + au_x = 0$, where a is constant, is not total variation diminishing, if $|a|\frac{k}{h} < 1$.
Hint: Check the total variation after one time step for a discontinuity.

b) Suppose the inviscid Burgers' equation in the form

$$u_t + \frac{1}{3}\left(uu_x + (u^2)_x\right) = 0\,, \qquad 0 \le x \le 1\,,$$

with periodic boundary conditions is discretized in space by the central difference operator D_0. The resulting semidiscretization becomes

$$(v_j)_t + \frac{1}{3}\left(v_j D_0 v_j + D_0(v_j^2)\right) = 0\,, \tag{7}$$

where $v_j = v_j(t)$ is continuous in t and approximates the exact solution $u(x_j, t)$.
Show that the energy $||v||_h^2 = h\sum_{j=1}^{N} v_j^2$ is not increasing, i.e. $\frac{d}{dt}||v||_h^2 \le 0$, and conclude that the semidiscretization (7) is stable.

(6p)

5. Consider the linear system $\mathbf{Au} = \mathbf{f}$ arising from discretizing the ODE

$$\begin{cases} -u'' = f(x)\,, & 0 \le x \le 1 \\ u(0) = u(1) = 0 \end{cases} \tag{8}$$

by $-D_+D_-u_j = f_j$, $j = 1, ..., n$, with $h = \frac{1}{n+1}$.

a) We know that the eigenvalues of $\mathbf{A}$ are given by $\lambda_\mu = \frac{4}{h^2}\sin^2(\frac{\mu\pi h}{2})$, $\mu = 1, \ldots, n$. Determine the eigenvalues of the iteration matrix of the damped Jacobi method

$$\mathbf{u}^{(m+1)} = \mathbf{u}^{(m)} - \frac{1}{2}\mathbf{D}^{-1}(\mathbf{A}\mathbf{u}^{(m)} - \mathbf{f})\,. \tag{9}$$

We take n odd and define $n_1 = n$, $h_1 = h$, $n_0 = \frac{n_1-1}{2}$, $h_0 = 2h_1$. Investigate how error modes with high wave numbers $\mu' = n_1 + 1 - \mu$ are damped by the damped Jacobi method compared to error modes with low wave numbers $\mu = 1, \ldots, n_0$.
Explain how low wave number error modes on the fine grid can become high wave number error modes on coarser grids in the multigrid method.
Use the findings in this task to explain why the multigrid method works so efficiently.

b) $-log_{10}(\rho(\mathbf{G}))$ is called the asymptotic convergence rate of an iterative method

$$\mathbf{u}^{(m+1)} = \mathbf{G}\mathbf{u}^{(m)} + \mathbf{d}\,. \tag{10}$$

Show that its reciprocal approximates the number of iterations to reduce the error by one decimal digit. How many iterations are sufficient to reduce the error by one decimal digit? Make a qualified guess on how many multigrid iterations are sufficient to reduce the error by one decimal digit. State your assumptions.

(6p)

Good luck!