

- Time: 9⁰⁰ – 14⁰⁰
- Tools: Pocket calculator, Beta Mathematics Handbook
- Maximum number of points: 40
- All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

Uppgift 1

- a) A and B are similar if $A = CBC^{-1}$ for some nonsingular matrix C .

$$\begin{aligned}
 Ax &= \lambda x \\
 &\Leftrightarrow \\
 C^{-1}Ax &= \lambda C^{-1}x \\
 &\Leftrightarrow \\
 \underbrace{C^{-1}AC}_B \underbrace{C^{-1}x}_y &= \lambda \underbrace{C^{-1}x}_y \\
 &\Leftrightarrow \\
 By &= \lambda y
 \end{aligned}$$

where $y = C^{-1}x$, i.e. B has eigenvalues λ and eigenvectors $C^{-1}x$ where λ are eigenvalues and x are eigenvectors of A .

- b) P Hermitian:

$$\begin{aligned}
 P^H &= (I - 2ww^H)^H = I^H - (2ww^H)^H = \\
 &= I - 2(ww^H)^H = I - 2(w^H)^H w^H = \\
 &= I - 2ww^H.
 \end{aligned}$$

P unitary:

$$\begin{aligned}
 P^H P &= PP = (I - 2ww^H)(I - 2ww^H) = \\
 &= I - 2ww^H - 2ww^H + 4ww^H ww^H = \\
 &= I - 4ww^H + 4w(w^H w)w^H = \\
 &= I - 4ww^H + 4ww^H = I.
 \end{aligned}$$

- c) Let $A = QR$ be a QR -factorisation. Since multiplication with a unitary matrix does not change the 2-norm,

$$\|Ax - b\|_2 = \|Q^H(QRx - b)\|_2 = \|Rx - Q^H b\|_2.$$

The matrix R is upper-triangular, i.e. we obtain the least-squares solution by solving the linear system of equations $R_1 x_1 = c_1$ where R_1 is the n first rows of R , x_1 is the n first rows of x and c_1 is the n first rows of $Q^H b$. Here A is an $m \times n$ -matrix.

Uppgift 2

- a) For an arbitrary $g = \sum_{i=1}^n d_i e_i \in M$, consider

$$\begin{aligned} \|f - g\|^2 &= \left\| f - \sum_{i=1}^n d_i e_i \right\|^2 = \left(f - \sum_{i=1}^n d_i e_i, f - \sum_{i=1}^n d_i e_i \right) \\ &= (f, f) - 2 \sum_{i=1}^n d_i (f, e_i) + \sum_{i=1}^n \sum_{j=1}^n d_i d_j (e_i, e_j) \\ &= \|f\|^2 + \sum_{i=1}^n (d_i - (f, e_i))^2 - \sum_{i=1}^n (f, e_i)^2. \end{aligned}$$

We see that $\|f - g\|$ is minimized when $d_i = (f, e_i)$, that is, for $g^* = \sum_{i=1}^n (f, e_i) e_i$.

- b) Since

$$(f - g^*, e_k) = \left(f - \sum_{i=1}^n (f, e_i) e_i, e_k \right) = (f, e_k) - \sum_{i=1}^n (f, e_i) (e_i, e_k) = (f, e_k) - (f, e_k) = 0,$$

for $k = 1, \dots, n$, the error $f - g^*$ is orthogonal to M .

Uppgift 3

- a) The largest eigenvalue is estimated by Gerschgorin's method. The eigenvalue is located in the circle with center -8 and radius $0.8 + 0.2 = 1$.
- b) Let the eigenvalues be ordered $|\lambda_1| > |\lambda_2| > |\lambda_3| > |\lambda_4|$. The convergence rate with the power method is $|\lambda_2|/|\lambda_1| \approx 4/8 = 0.5$. Since $0.5^{10} \approx 0.001$, about 10 iterations are needed.
- c) Use the inverse power method. Let

$$\begin{aligned} y^k &= z^k / \|z^k\|_2, \\ Az^{k+1} &= y^k. \end{aligned} \tag{1}$$

This is the power method applied to A^{-1} with the largest eigenvalue $1/0.3041$. Hence, the iterations converge to that eigenvalue λ_4^{-1} , the inverse of the smallest eigenvalue of A .

- d) Use the shifted inverse power method with the shift 4. Let A in the algorithm (1) be equal to $A - 4I$.

Uppgift 4

a) The approximation

$$h^{-2}(-u_{i+1} + 2u_i - u_{i-1}) + ihu_i = f_i, \quad i = 1, 2, \dots, N-1,$$

with $u_0 = 1$ and $u_N = 0$ is second order accurate.

b) The matrix A is

$$A = \begin{pmatrix} 2+h^3 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2+2h^3 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2+3h^3 & -1 & 0 & \dots \\ 0 & \dots & 0 & -1 & 2+(N-2)h^3 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2+(N-1)h^3 \end{pmatrix}.$$

The vector b is $(1 + h^2 f_1, h^2 f_2, \dots, h^2 f_{N-1})^T$.

c) The iteration is

$$u^{k+1} = Mu^k + c = D^{-1}Cu^k + D^{-1}b,$$

where

$$D = \begin{pmatrix} 2+h^3 & 0 & 0 & \dots & \dots & 0 \\ 0 & 2+2h^3 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2+3h^3 & 0 & 0 & \dots \\ 0 & \dots & 0 & 0 & 2+(N-2)h^3 & 0 \\ 0 & \dots & \dots & 0 & 0 & 2+(N-1)h^3 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & \dots & 0 & 1 & 0 & 1 \\ 0 & \dots & \dots & 0 & 1 & 0 \end{pmatrix}.$$

d) The eigenvalues of M are estimated by Gerschgorin's circles. The eigenvalues are in the union of the circles with center at 0 with the radii $2/(2 + jh^3)$. The radii are all less than 1 and therefore the eigenvalues of M are less than 1 in magnitude and the Jacobi method converges.

Uppgift 5

$$\begin{aligned}
 \text{a)} \quad (u, D_+ v)_{r,s} &= \sum_{j=r}^s u_j^T (D_+ v)_j h = \sum_{j=r}^s u_j^T (v_{j+1} - v_j) = \sum_{j=r+1}^{s+1} u_{j-1}^T v_j - \sum_{j=r}^s u_j^T v_j \\
 &= \sum_{j=r+1}^{s+1} u_{j-1}^T v_j - \sum_{j=r+1}^{s+1} u_j^T v_j + u_{s+1}^T v_{s+1} - u_r^T v_r \\
 &= - \sum_{j=r+1}^{s+1} (u_j - u_{j-1})^T v_j + u_{s+1}^T v_{s+1} - u_r^T v_r \\
 &= -(D_- u, v)_{r+1, s+1} + u_{s+1}^T v_{s+1} - u_r^T v_r \\
 (u, Av)_{r,s} &= \sum_{j=r}^s u_j^T A v_j h = \sum_{j=r}^s u_j^T A^T v_j h = \sum_{j=r}^s (A u_j)^T v_j h = (A u, v)_{r,s}
 \end{aligned}$$

$$\begin{aligned}
 \text{b)} \quad \frac{d}{dt} \|u\|_{1,N-1}^2 &= 2(u, u_t)_{1,N-1} = 2(u, A D_+ D_- u)_{1,N-1} = 2(A u, D_+ D_- u)_{1,N-1} \\
 &= -2(D_- A u, D_- u)_{2,N} + (A u)_N^T (D_- u)_N - (A u)_1^T (D_- u)_1 \\
 &= -2 \sum_{j=2}^N (D_- A u)_j^T (D_- u)_j h + 0 - (A u)_1^T u_1 h^{-1} \\
 &= -2 \sum_{j=2}^N (D_- u)_j^T A^T (D_- u)_j h - u_1^T A^T u_1 h^{-1} \leq 0
 \end{aligned}$$

where the last inequality follows from the fact that A is symmetric positive definite and the last equality follows from that A and D_- commute. Hence

$$\|u(\cdot, t)\|_{1,N-1} \leq \|u(\cdot, 0)\|_{1,N-1} = \|f(\cdot)\|_{1,N-1},$$

which shows that the approximation is stable with the norm $\|u\|_{1,N-1}$.