



Solutions to Exam 2007-10-19

- 1 (a) From the problem formulation we know that λ is an eigenvalue to the matrix A and that x is the corresponding eigenvector.

Multiplying $Ax = \lambda x$ from the left by A repeatedly we get that

$$A^2x = \lambda Ax = \lambda^2 x$$

$$A^3x = \lambda^2 Ax = \lambda^3 x$$

⋮

$$A^m x = \lambda^m x.$$

From this we see that λ^m is an eigenvalue to A^m and that the corresponding eigenvector is x .

(b)

$$C = \begin{pmatrix} 3 & 4 \\ 7 & 2 \end{pmatrix} \Rightarrow C^T = \begin{pmatrix} 3 & 7 \\ 4 & 2 \end{pmatrix} \Rightarrow$$

$$C^T C = \begin{pmatrix} 58 & 26 \\ 26 & 20 \end{pmatrix}$$

$$\begin{vmatrix} 58 - \lambda & 26 \\ 26 & 20 - \lambda \end{vmatrix} = (58 - \lambda)(20 - \lambda) - 676 = 0$$

$$\lambda^2 - 78\lambda + 484 = 0 \Rightarrow \lambda = 39 \pm \sqrt{1073}$$

Hence $\rho(C^T C) = 39 + \sqrt{1037} \approx 71.20$ and $\|C\|_2 = \sqrt{39 + \sqrt{1037}} \approx 8.44$.

(c)

$$A \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} = y$$

The normal equations read:

$$A^T A \begin{pmatrix} a \\ b \end{pmatrix} = A^T y \Rightarrow$$

$$\begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 45 \\ 20 \end{pmatrix} \Rightarrow$$



$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5/2 \\ 5/3 \end{pmatrix},$$

i.e. $y = \frac{5}{2}x + \frac{5}{3}$.

2 (a) A Taylor expansion gives the local truncation error

$$\begin{aligned} & \frac{u(x, t+k) - (u(x+h, t) + u(x-h, t))/2}{k} - \alpha \frac{u(x+h, t) - u(x-h, t)}{2h} = \\ &= \dots = \frac{k}{2}u_{tt} - \frac{h^2}{2k}u_{xx} - \frac{\alpha h^2}{6}u_{xxx} + \dots \end{aligned}$$

since $u_t = \alpha u_x$.

(b) The discrete Fourier transform of the difference equation (using orthogonality of $e^{i2\pi mk/N}$) is

$$\frac{\hat{u}_m^{n+1} - \cos(2\pi m/N) \hat{u}_m^n}{k} = \alpha \frac{i \sin(2\pi m/N)}{h} \hat{u}_m^n$$

or equivalently $\hat{u}_m^{n+1} = \hat{g}_m \hat{u}_m^n$, where

$$\hat{g}_m = \cos \frac{2\pi m}{N} + i \frac{k\alpha}{h} \sin \frac{2\pi m}{N}.$$

Thus,

$$|\hat{g}_m|^2 = \cos^2 \frac{2\pi m}{N} + \frac{k^2 \alpha^2}{h^2} \sin^2 \frac{2\pi m}{N} \leq 1 \quad \forall m,$$

(i.e. the scheme is stable) if and only if

$$\frac{k^2 \alpha^2}{h^2} \leq 1 \quad \Leftrightarrow \quad \frac{k}{h} \leq \frac{1}{|\alpha|}.$$

3 Let $A = D + L + U$, where D is the diagonal part of A , L is the lower triangular part of A and U is the upper triangular part of A .

(a) Jacobi: $x^{k+1} = -D^{-1}(L+U)x^k + D^{-1}b$,
 Gauss-Seidel: $x^{k+1} = -(D+L)^{-1}Ux^k + (D+L)^{-1}b$.

(b) Jacobi: $x = -D^{-1}(L+U)x + D^{-1}b \Rightarrow (D+L+U)x = b$, i.e. $Ax = b$.
 Gauss-Seidel: $x = -(D+L)^{-1}Ux + (D+L)^{-1}b \Rightarrow (D+L+U)x = b$, i.e. $Ax = b$.

(c)

$$A = \begin{pmatrix} 4 & 2 & 0 \\ -3 & 5 & -1 \\ 0 & -1 & 4 \end{pmatrix} \Rightarrow$$

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$



$$x^{k+1} = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.25 \end{pmatrix} \left(- \begin{pmatrix} 0 & 2 & 0 \\ -3 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) + \\ \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.25 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 2.0 \\ 2.5 \end{pmatrix}$$

4 (a) The first column is already, i.e., $P_1 = I$. We then get

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p_{11} & p_{12} \\ 0 & p_{21} & p_{22} \end{pmatrix}$$

where $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ is computed with a Householder matrix $P = I - 2ww^T$. Here

$$P = \begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}$$

which gives

$$R = QA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & 15.8 \\ 0 & 0 & -19.4 \end{pmatrix}$$

(b) Repeat until convergence:

$$A \Rightarrow QR \quad \text{Factorize}$$

$$A \Leftarrow RQ \quad \text{Multiply}$$

A becomes block overtriangular where the diagonal blocks are 1x1 including the real eigenvalues and 2x2 including complex pairs of eigenvalues. One iteration (QR-factorization as above):

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 5 & 15.8 \\ 0 & 0 & -19.4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \\ 0 & 0.6 & -0.8 \end{pmatrix} = \begin{pmatrix} 1 & 0.6 & -0.8 \\ 0 & 13.48 & -9.64 \\ 0 & -11.64 & 15.52 \end{pmatrix}$$

5 Construct the residual $r(x) = u'' + a \cdot u - f(x)$ and require that

$$\int_{-1}^1 r(x)v(x)dx = 0, \quad \text{for all } v(x) \in V \text{ with } v(-1) = v(1) = 0$$

Integrate by parts to lower the regularity requirements:

$$\int_{-1}^1 u''v dx = [u'v]_{-1}^1 - \int_{-1}^1 u'v' dx = - \int_{-1}^1 u'v' dx$$



$$\Rightarrow - \int_{-1}^1 u' v' dx + \int_{-1}^1 a u v dx = \int_{-1}^1 f v dx \quad \text{for all } v(x) \in V$$

Set $u(x) = \sum_i c_i \varphi_i(x)$ and let $\{\varphi_j(x)\}_j$ be a basis in V .

$$\Rightarrow \sum_i c_i \left(- \int_{-1}^1 \varphi'_i \varphi'_j dx + \int_{-1}^1 a \varphi_i \varphi_j dx \right) = \int_{-1}^1 f \varphi_j dx \quad \forall j$$

The variational formulation gives a linear system of equations with the stiffness matrix

$$A_{ij} = - \int_{-1}^1 \varphi'_i \varphi'_j dx + a \int_{-1}^1 \varphi_i \varphi_j dx$$

Basis functions

(a) φ_k = piecewise linear

$$\begin{cases} \int_{-1}^1 \varphi_k \varphi_l dx = 0, & \text{om } |k - l| > 1 \\ \int_{-1}^1 \varphi'_k \varphi'_l dx = 0, & \text{om } |k - l| > 1 \end{cases}$$

\Rightarrow tridiagonal matrix (non-overlapping basis functions if $|k - l| > 1$).

(b) $\varphi_k = \sin(k\pi x)$

$$\begin{cases} \int_{-1}^1 \varphi_k \varphi_l dx = 0, & \text{om } k \neq l \\ \int_{-1}^1 \varphi'_k \varphi'_l dx = 0, & \text{om } k \neq l \end{cases}$$

\Rightarrow diagonal matrix (orthogonal basis functions).

The second system becomes easier to solve.