Uppsala Universitet
Insitutionen för informationsteknologi
Avdelningen för teknisk databehandling

2007-01-16
Scientific Computing II/NV2

Time: $9^{00} - 14^{00}$
Tools: Pocket calculator, Beta Mathematics Handbook
Maximum number of points: 40

All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

1. a) Consider the matrix

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 3 & -7 \\ 1 & 4 & 6 \end{pmatrix}.$$

Compute $\|A\|_\infty$ and $\|A\|_1$. (2p.)
b) Consider the matrix

$$B = \begin{pmatrix} 2 & 8 \\ 6 & 9 \end{pmatrix}.$$

Compute $\rho(B)$. (2p.)
c) Define a Householder-matrix P. Show that $P = P^T$. (4p.)

2. a) We are interested in solving a linear system of equation $Ax = b$. One way to do this is to use Gaussian elimination. However, if the matrix A is large and sparse this is not a good idea. Describe why this isn't a good idea and suggest a better way to solve $Ax = b$ in this case. You should also motivate why the suggested method is better. (4p.)
b) The linear system of equations $Bu = b$ where

$$B = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{pmatrix}$$

and $b = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T$ should be solved using Jacobi's method. Perform one iteration using the initial guess $u^0 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$. (2p.)
c) Show that the iteration in b) will converge. (2p.)

3. A signal y was measured at $N = 5$ times $t_i = \frac{i}{N}$, $i = 0, \ldots, N-1$. We know that the signal can be described with a combination of certain sine and cosine functions, but we don't know how much each function contributes. We will find this out using a least squares approximation in the linear subspace $M = \mathrm{span}\{\sin(2\pi t), \cos(2\pi t)\}$. Define the scalar product as $(f, g) = \sum_{i=0}^{N-1} f(t_i)g(t_i)$.

   a) Determine an ON-basis in M. As a help, you may consider $(\sin(2\pi t), \cos(2\pi t)) = 0$ as given. (2p.)

   b) Find the best approximation to the following measured signal in the least squares sense.

t_i	0	0.2	0.4	0.6	0.8
y_i	-3.0	0.0	3.0	1.8	-1.8

   Give your answer on the form $a_1 \sin(2\pi t) + a_2 \cos(2\pi t)$. (2p.)

   c) Formulate the problem as an overdetermined system of equations

$$Ax = b.$$

   (2p.)

   d) Name (or describe) one method to compute a QR-factorization of A. Describe how the QR-factorization is used for solving the overdetermined system $Ax = b$. (2p.)

4. Consider the matrix

$$A = \begin{bmatrix} 1 & -0.3 & a \\ -0.2 & 2 & 0.3 \\ b & 0.1 & 3 \end{bmatrix},$$

   where a and b are real numbers.

   a) Give the largest possible bound on the absolute values of a and b for which Gershgorin's theorem guarantees that the eigenvalues of A are distinct. (4p.)

   b) Using that $\lambda_1 \approx 1$, $\lambda_2 \approx 2$ and $\lambda_3 \approx 3$, give an estimate of the number of iterations that the power method would need to compute the eigenvector v_3 of λ_3 with an error less than 10^{-5}, given that the initial error is of order 1. (2p.)

   c) Let $a = b = 0$. Suggest an iterative method to compute the eigenvector of the middle eigenvalue $\lambda_2 \approx 2$. (2p.)

5. The PDE

$$\begin{aligned} &u_t + au_x = 0,\ 0 < x < 1,\ t > 0, \\ &u(x,0) = g(x),\ u(0,t) = u(1,t), \end{aligned} \tag{1}$$

is approximated on the grid $x_j = jh, j = 0, \dots, N-1$, by the forward Euler method in time with the time step k.
a) The space derivative is approximated by

$$u_x(x_j, t_n) \approx (u^n_{j+1} - u^n_{j-1})/(2h).$$

For which a is the discretization of the PDE stable? (3p.)
b) The space derivative is approximated by

$$u_x(x_j, t_n) \approx (u^n_{j+1} - u^n_j)/h.$$

For which a is the discretization of the PDE stable? (5p.)

Good luck! Lina, Per and Elisabeth.