## UPPSALA UNIVERSITY

Department of Information Technology
Division of Scientific Computing
Exam in Scientific Computing II/NV2 2007-05-22

- Time: $14^{00}-19^{00}$
- Tools: Pocket calculator, Beta Mathematics Handbook
- Maximum number of points: 40
- All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.


## Uppgift 1

Last year, a local strawberry producer measured the output of her strawberry fields in the weeks around midsummer. The result was the following

$$
\begin{array}{r|rrrr}
x_{i} & -1 & -1 / 3 & 1 / 3 & 1 \\
\hline y_{i} & 0 & 1 & 3 & 3
\end{array}
$$

where $x$ is the number of weeks relative to midsummer and $y$ is the daily strawberry output in $\left[\ell / m^{2}\right]$. This year, she wants to use the measurement to optimize the profit. However, she needs a day by day estimate. From experience, she has reason to believe that the data can be well approximated by a second order polynomial. Let $M=\operatorname{span}\left\{1, x, x^{2}\right\}$ and use the scalar product $(f, g)=\sum_{i=1}^{4} f\left(x_{i}\right) g\left(x_{i}\right)$.
a) Determine an ON-basis in $M$.
b) Compute a least squares approximation $g(x)=a+b x+c x^{2}$ to the given data. [4p]

## Uppgift 2

A group of students have been asked to implement an iterative method for computing the largest eigenvalue of a matrix $A$ and the corresponding eigenvector. Depending on the matrix, the implementation should choose either the power method or inverse iteration with a suitable shift.
a) Consider the following matrix

$$
A=\left[\begin{array}{rrr}
0.5 & 0.2 & -0.3 \\
0.2 & -2.0 & 0.0 \\
-0.3 & 0.0 & -0.5
\end{array}\right]
$$

and the two choices

- the power method with shift $A+1.25 I$, or
- inverse iteration with shift $A-0.5 I$.

Determine which of the two methods is likely to require the smallest number of iterations. Use Gersgorin's theorem to estimate the worst case for the error and convergence rate for each method.
b) For large matrices $A$ the computational cost for each method must also be considered carefully. Let $A$ be a full (not sparse) $N \times N$ matrix. Use the estimates below to give an expression for the cost of each method assuming that the power method needs $k_{p}$ iterations and inverse iteration requires $k_{i}$ iterations to converge. Any cost not given in the table can be neglected. What should the relation between $k_{p}$ and $k_{i}$ be for the two methods to be equally fast?

| Action | Cost |
| :--- | :--- |
| Normalizing a vector to have unit norm | $3 N$ |
| Estimating the eigenvalue (a scalar product) | $2 N$ |
| Computing the LU-factorization of a matrix $A$ | $2 N^{3} / 3$ |
| Solving the system $A z=y$ using the LU-factorization | $2 N^{2}$ |
| Multiplying a vector with a matrix $A$ | $2 N^{2}$ |

## Uppgift 3

Let the matrix $B$ be defined by

$$
B=\left[\begin{array}{cccc}
-0.1 & 0.2 & -0.15 & 0.65 \\
0.25 & 0.5 & 0.1 & -0.1 \\
-0.15 & -0.1 & 0.2 & -0.1 \\
0.3 & 0.05 & 0.15 & 0.1
\end{array}\right]
$$

a) Compute $\|B\|_{1}$ and $\|B\|_{\infty}$.
b) Consider the iteration $x^{k+1}=B x^{k}+d$ where, $d=\left[\begin{array}{llll}0 & 0.1 & 0.3 & -0.2\end{array}\right]^{T}$ and $x^{0}=\left[\begin{array}{llll}0.1 & 0.2 & -0.1 & 0.4\end{array}\right]^{T}$. Compute $x^{1}$.
c) Show that the iteration in b) will converge. You can consider as known that the spectral radius $\rho(B) \leq\|B\|$, for all norms.
d) Now, consider the linear system of equations

$$
\begin{equation*}
B u=b, \tag{1}
\end{equation*}
$$

where $b=\left[\begin{array}{cccc}0.2 & 1.0 & -1.2 & -1.0\end{array}\right]^{T}$. Define Jacobi's method to solve (1) on the form $x^{k+1}=M x^{k}+c$. You shall compute all entries in $M$ and $c$.

## Uppgift 4

We are interested in solving the following partial differential equation (PDE)

$$
\begin{cases}u_{t}=\lambda u_{x x}, & \lambda>0, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T  \tag{2}\\ u(x, 0)=f(x), \\ u(0, t)=u(1, t), & \text { periodic boundary conditions }\end{cases}
$$

a) Of what type is the PDE in (2)?
b) Equation (2) is discretized on the grid $x_{j}=j \cdot h, t_{n}=n \Delta t, j=0, \ldots, M, n=0, \ldots, N$, $h=1 / M$ and $\Delta t=T / N$ as

$$
\begin{cases}\frac{u_{j}^{n+1}-u_{j}^{n}}{\Delta t}=\lambda \frac{u_{j+1}^{n}-2 u_{j}^{n}+u_{j-1}^{n}}{h^{2}}, & j=0, \ldots, M, \quad n=0, \ldots, N-1  \tag{3}\\ u_{j}^{0}=f_{j}, & j=0, \ldots, M, \\ u_{j}^{n}=u_{j+M}^{n}, & \forall j, n=0, \ldots, N-1\end{cases}
$$

Define the solution vector as

$$
u^{n+1}=\left[\begin{array}{c}
u_{0}^{n+1} \\
u_{1}^{n+1} \\
\vdots \\
u_{M-1}^{n+1}
\end{array}\right]
$$

and set up (3) on the form $u^{n+1}=A u^{n}$.
c) Derive the local truncation error when (3) is used for approximating (2).
d) Derive a stability condition for (3).

## Uppgift 5

Consider the problem

$$
\begin{align*}
& -u^{\prime \prime}=f \quad \text { on }(0,1),  \tag{4}\\
& a u(0)+u^{\prime}(0)=1, \quad a u(1)+u^{\prime}(1)=1 . \tag{5}
\end{align*}
$$

where $a>0$ is constant.
a) Derive the variational formulation of problem $(4,5)$.

Hint: Letting $\mathcal{V}=\left\{u \mid u\right.$ and $u^{\prime}$ square integrable $\}$ the variational formulation reads:
Find $u \in \mathcal{V}$ such that

$$
\int_{0}^{1} u^{\prime} v^{\prime} d x-v(1)(1-a u(1))+v(0)(1-a u(0))=\int_{0}^{1} f v d x \quad \forall v \in \mathcal{V}
$$

b) Now, change the boundary conditions to

$$
\begin{equation*}
u(0)=u(1)=0 . \tag{6}
\end{equation*}
$$

Define a finite element method for problem (4,6) using piecewise linear (hat) functions and show that the corresponding matrix is symmetric and positive definite.
c) For the boundary conditions (6) show that the solution is bounded in terms of the data of the problem, that is

$$
\|u\|_{L^{2}(0,1)}+\left\|u^{\prime}\right\|_{L^{2}(0,1)} \leq C\|f\|_{L^{2}(0,1)},
$$

where $C$ is a constant and

$$
\begin{equation*}
\|v\|_{L^{2}(0,1)}=\left(\int_{0}^{1} v^{2} d x\right)^{1 / 2} . \tag{3p}
\end{equation*}
$$

Hint: $u(x)=\int_{0}^{x} u^{\prime}(y) d y$.

