UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing
Exam in Scientific Computing II/NV2 2007-08-21

- Time: $9^{00} - 14^{00}$
- Tools: Pocket calculator, Beta Mathematics Handbook
- Maximum number of points: 40
- All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

Uppgift 1

a) Show that similar matrices have the same eigenvalues but different eigenvectors. [3p.]

b) The matrix $P = I - 2ww^H$ where $w^H w = 1$ is a so called Householder-matrix. Show that P is hermitian and unitary. [2p.]

c) Show how we can solve the overdetermined system of equations $Ax = b$ in a least-square sense by using a QR-faktorisation of A. [3p.]

Uppgift 2

a) Let L be a linear space with inner product (f, g) and norm $\|f\| = \sqrt{(f, f)}$. Also let M be a linear subspace to L, i.e. $M \subset L$. A least squares problem can then be formulated as given $f \in L$, derive $g^* \in M$ that minimizes $F(g) = \|f - g\|$. Now define an ON-basis $\{e_i\}_{i=1}^n$ in M. Show that the least squares problem has a unique solution $g^* = \sum_{i=1}^n (f, e_i)e_i$. [5p.]

b) Show that the error $r = f - g^*$ is orthogonal to M. [3p.]

Uppgift 3

The matrix

$$A = \begin{pmatrix} -1 & 2 & 0 & 0 \\ -3 & 5 & -1 & 0 \\ 0 & 0.8 & -8 & 0.2 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

has three eigenvalues $0.3041, 3.6459, 4.0255$.

a) The fourth eigenvalue has the largest magnitude. Estimate that eigenvalue. [2p.]

b) The eigenvalue with largest magnitude is computed with the power method. How many iterations do we need to reduce the initial error by a factor 1000? [2p.]

c) Describe how the smallest eigenvalue can be computed by a modification of the power method and explain why the iterations converge. [2p.]

d) Describe how the eigenvalue close to 4 can be computed by a modification of the power method. [2p.]

Uppgift 4

We wish to solve the boundary value problem

$$\begin{aligned} -u_{xx} + xu &= f, \quad 0 < x < 1, \\ u(0) = 1, &\quad u(1) = 0, \end{aligned} \tag{1}$$

on the grid $x_i = ih,\ i = 0, \dots, N$, with $h = 1/N$.

a) Suggest a second order finite difference approximation of the differential equation (1). [2p.]

b) The solution $u_i,\ i = 1, \dots, N-1$, satisfies a system of linear equations

$$Au = b. \tag{2}$$

What are the elements of A and b? [2p.]

c) The system (2) is solved by the Jacobi method. Write in detail the iteration formula for each iteration with the method. [2p.]

d) Show that the iterations converge. [2p.]

Uppgift 5

Consider the problem

$$\begin{aligned} u_t &= Au_{xx}, & x \in (0,1), \quad & t \geq 0, \\ u(0,t) &= u(1,t) = 0, & & t \geq 0, \\ u(x,0) &= f(x), & x \in (0,1), & \end{aligned} \tag{3}$$

where u is a vector function and A is symmetric and postive definite matrix. A semi-discretized version is obtained using centered differences on the spacial grid $x_j = jh$, $j = 0, 1, \dots, N$, with $x_N = 1$.

$$\begin{aligned} \frac{\mathrm{d}u_j}{\mathrm{d}t} &= AD_+D_-u_j, & j &= 1, 2, \dots, N-1 \\ u_0(t) = u_N(t) &= 0 & t &\geq 0, \\ u_j(0) &= f(x_j) & j &= 1, 2, \dots, N-1. \end{aligned} \tag{4}$$

We define the scalar products

$$(u, v)_{r,s} = \sum_{j=r}^{s} u_j^T v_j h,$$

and denote the corresponding norms $\|u\|_{r,s}$, that is

$$\|u\|_{r,s}^2 = (u, u)_{r,s}.$$

a) Show that

$$\begin{aligned} (u, D_+v)_{r,s} &= -(D_-u, v)_{r+1,s+1} + u_{s+1}^T v_{s+1} - u_r^T v_r \\ (u, Av)_{r,s} &= (Au, v)_{r,s} \end{aligned}$$

[3p.]

b) Prove that (4) is stable in the norm $\|u\|_{1,N-1}$, that is,

$$\|u(\cdot, t)\|_{1,N-1} \leq \|f(\cdot)\|_{1,N-1} \quad t > 0.$$

[5p.]

Good luck!
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