

## Scientific Computing II/NV2, 2007-10-19

**Time:** 8<sup>00</sup> – 13<sup>00</sup>

**Help:** Pocket calculator, Beta Mathematics Handbook

All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

1. a) Let  $\lambda$  be an eigenvalue to a matrix  $A$  and  $x$  the corresponding eigenvector. Show that  $\lambda^m$  is an eigenvalue to  $A^m$ . What is the corresponding eigenvector? (2p)

- b) Consider the matrix  $C = \begin{pmatrix} 3 & 4 \\ 7 & 2 \end{pmatrix}$ . Compute  $\|C\|_2 = \sqrt{\rho(C^T C)}$ . (2p)

- c) The following data has been collected

x	1	2	3
y	4	7	9

Compute the straight line  $y = ax + b$  that approximates the data best in a least squares sense when  $(f, g) = \sum_{i=1}^3 f(x_i)g(x_i)$  and  $\|f\| = \sqrt{(f, f)}$ . (4p)

2. The differential equation

$$\begin{cases} u_t = \alpha u_x, & t > 0, 0 < x < 1, \\ u(x, 0) = f(x), & 0 \leq x \leq 1, \\ u(0, t) = u(1, t), & t \geq 0, \text{ periodic boundary conditions} \end{cases}$$

where  $\alpha$  is a real constant is approximated by

$$\begin{cases} \frac{u_j^{n+1} - (u_{j+1}^n + u_{j-1}^n)/2}{k} = \alpha \frac{u_{j+1}^n - u_{j-1}^n}{2h}, & j = 1, \dots, N, n = 0, \dots, \\ u_j^0 = f(x_j), & j = 1, \dots, N, \\ u_j^n = u_{j+N}^n, & \forall j, n = 0, \dots \end{cases}$$

$k$  is the time-step and  $h = 1/N$ .

- a) Derive the local truncation error. (4p)  
 b) Show that the scheme is stable for  $k/h \leq 1/|\alpha|$ . (4p)

3. A system of equations

$$Ax = b$$

shall be solved with an iterative method

$$x^{k+1} = Mx^k + c.$$

- a) For a general  $A$  and  $b$ , tell what  $M$  and  $c$  look like for the Jacobi and Gauss-Seidel method. (2p)

- b) Show that the fixpoint  $x = Mx + c$  fulfills  $x = A^{-1}b$  (i.e. solves  $Ax = b$ ) for both the Jacobi and Gauss-Seidel method. (2p)

- c) The system of equations

$$\begin{pmatrix} 4 & 2 & 0 \\ -3 & 5 & -1 \\ 0 & -1 & 4 \end{pmatrix} x = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

shall be solved with the Jacobi method. Perform one iteration with Jacobis method when

$$x^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(4p)

4. We are interested in computing all eigenvalues of

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & 3 & 25 \end{pmatrix}$$

- a) Compute the QR-factorization of  $A$  using Householder-transformations. (4p)  
 b) Describe how you would use the QR-method to compute all eigenvalues and perform one iteration using your QR-factorization from above. (4p)

5. Consider the following boundary value problem

$$\begin{aligned} u'' + au &= f, & -1 < x < 1, \\ u(-1) &= 0, \\ u(1) &= 0, \end{aligned}$$

where  $a$  is a real constant.

- a) Derive the variational formulation of the differential equation. (3p)  
 b) Define the piecewise linear hat functions  $\{\phi_i(x)\}_{i=1}^N$  by

$$\phi_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases},$$

where  $x_j = -1 + 2j/(N+1)$ ,  $j = 0, \dots, N+1$ . Approximate the solution as  $u_h = \sum_{i=1}^N c_i \phi_i(x)$  for some coefficients  $c_i$ . Set up and derive the linear system  $Ac = b$  for the coefficients  $c_i$  according to the variational formulation. You don't have to compute the actual elements in the matrix, you can answer in integral form, but you should indicate which elements are non-zero and give an expression how these can be computed (answer on the form  $A_{ij} = \int_{-1}^1 (\dots) dx$ ). (3p)

- c) Change the basis functions to  $\phi_k(x) = \sin(k\pi x)$  and derive the matrix  $A$  (in integral form as above) for the coefficients  $c_i$  using these basis functions. Be sure to indicate which elements are non-zero and which are zero. Which of the linear systems above requires least work to solve? (2p)

Good Luck! Lina and Jarmo