Fourier Transforms and the Frequency Domain

Lecture 11

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Reading Instructions and Assignment
Chapters and Assignment for This Lecture

- Chapters 4.3 – 4.6 in Gonzales-Woods.
- There is no assignment for this lecture.
Ideal Lowpass Filters and “Ringing”

In the frequency domain, define a cut-off radius $D_0$.

**Ideal Lowpass Filter (ILPF)**

$$H(u, v) = \begin{cases} 
1 & \text{if } D(u, v) \leq D_0; \\
0 & \text{if } D(u, v) > D_0. 
\end{cases}$$

To find $h(x, y)$:

1. **Centering**: $H(u, v) \ast (-1)^{u+v}$
2. **Inverse Fourier transform (FT)**
3. **Multiply real part by** $(-1)^{x+y}$
Ideal Lowpass Filters and “Ringing”

Properties of $h(x, y)$:

1. It has a central dominant circular component (providing the blurring)
2. It has concentric circular components (rings) giving rise to the ringing effect.

Example of $h(x)$ from the inverse FT of a disc (ILPF) with radius 5.
Ideal Lowpass Filters and “Ringing”

Original image (top left) and filtered images with ILPF of radius 5, 15 and 30, removing 8, 5.4 and 3.6% of the total power. This type of artefacts are not acceptable in e.g. medical imaging.
Reduce “Ringing” with Non-Ideal Lowpass Filters

Butterworth Lowpass Filter (BLPF)

\[ H(u, v) = \frac{1}{1 + \left( \frac{D(u,v)}{D_0} \right)^{2n}} \]

- \( n \) is the order of the filter
- A high \( n \) will cause “ringing” (approaching ILPF)
- No sharp discontinuity
Butterworth Lowpass Filter

In general, BLPFs of order 2 are a good compromise between effective lowpass filtering and acceptable ringing characteristics.

BLPF “ringing” effects for different values of $n$. 

In general, BLPFs of order 2 are a good compromise between effective lowpass filtering and acceptable ringing characteristics.
Gaussian Lowpass Filter

Gaussian Lowpass Filter (GLPF)

\[ H(u, v) = e^{-\frac{D^2(u,v)}{2D_0^2}} \]

- \( D_0 \) is the standard deviation (\( \sigma \)), or the “spread” of the Gaussian.
- The inverse FT of a Gaussian is also a Gaussian, meaning a Gaussian smoothing in the spatial domain.
- Guarantees no ringing.
Highpass Filters

Ideal Highpass Filter (IHPF)

\[ H(u, v) = \begin{cases} 
0 & \text{if } D(u, v) \leq D_0; \\
1 & \text{if } D(u, v) > D_0.
\end{cases} \]

Butterworth Highpass Filter (BHPF)

\[ H(u, v) = \frac{1}{1 + \left( \frac{D_0}{D(u,v)} \right)^{2n}} \]

Gaussian Highpass Filter (GHPF)

\[ H(u, v) = 1 - e^{-\frac{D^2(u,v)}{2D_0^2}} \]

- Same ringing effect as for the lowpass filters.
Highpass Filters

Spatial representation of different highpass filters (ideal, Butterworth and Gaussian).

Ringing effect noticeable in ideal and Butterworth filters.
### Unsharp Masking

\[ f_{HP}(x, y) = f(x, y) - f_{LP}(x, y) \]

### High-Boost Filtering (general)

\[ f_{HB}(x, y) = (A - 1) \ast f(x, y) + f_{HP}(x, y) \quad \text{where} \quad A \geq 1 \]
\[ H_{HB}(u, v) = (A - 1) + H_{HP}(u, v) \]

### High-Frequency Emphasis

\[ H_{HFE}(u, v) = a + b \ast H_{HP}(u, v) \]
High-Boost Filtering

Result of high-boost filtering (original, highpass, $A = 2$ and $A = 2.7$).
High-Frequency Emphasis Filtering

Result of high-frequency emphasis filtering (original, highpass, high-freq emphasis and histogram equalization).
Periodicity and Padding

Consider image of size $M \times N$.

- **Periodicity:**
  \[
  F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N)
  \]

- **Symmetry:** $|F(u, v)| = |F(-u, -v)|$
Padding in the Spatial Domain (SD)

Padding

When introducing identical periods at convolution to avoid “wrap-around error”.
Padding in the Frequency Domain (FD)

Padding is important in the frequency domain as well.

- Convolution in SD is multiplication in FD.
- The filter and the image must have the same size at multiplication!

When padding, the image borders (if not black) become sharp edges, leading to “ringing” at the image borders after filtering using ideal filters.

The padding is removed after the filtering.
Correlation Function

\[ f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n) \]

- \( f^* \) is the complex conjugate of \( f \) (for images \( f^* = f \)).
- Positive summation; \( h \) not mirrored about origin.

Correlation Theorem

\[ f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v) H(u, v) \]
\[ f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \circ H(u, v) \]
There is a strong similarity between convolution and correlation.