Introduction to the Frequency Domain
Lecture 04

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Subjects

- Neighborhood relationships (adjacency, connectivity).
- Linear filtering (Averaging).
- Non-linear filtering (Min, Max, and Median).
- Edge filters (Sobel, Laplace, and edge enhancing).
Which filters have been used?
The Fourier Transform
Jean Baptiste Joseph Fourier (March 21, 1768 - May 16, 1830)

- French mathematician ("Théorie analytique de la chaleur", 1822).
- Fourier series: periodically repeating functions can be expressed as a sum of sines and/or cosines of different frequencies and weights.
- Fourier transform: Functions that are not periodic (but with finite area under the curve) can be expressed as the integral of sines and/or cosines of different frequencies and weights.
- Representation using Fourier series or transforms allows for complete recovery of the original function.
- Different way to represent the same information.
- Light & prism: Separation of wavelengths.
The Fourier Transform

- Frequency: rate of change
- High frequencies correspond to sharp edges, fine detail and noise.
- Low frequencies correspond to smoother and slower changes.

The function at the bottom is the sum of the four functions above it. Fourier’s idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.
The Fourier Transform

Continuous 1D and 2D case

The one-dimensional Fourier transform

\[ F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} \, dx \]

and its inverse

\[ f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} \, du. \]

Fourier transform pair in 2D,

\[ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi (ux + vy)} \, dx \, dy \]

and,

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi (ux + vy)} \, du \, dv. \]
The Fourier Transform

Discrete 1D case

- Discrete Fourier transform

\[ F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \text{ for } u = 0, 1, \ldots, M-1. \]

- Euler's formula

\[ e^{j\theta} = \cos(\theta) + j \sin(\theta). \]

- Substitution gives

\[ F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) \left[ \cos \left( \frac{2\pi ux}{M} \right) - j \sin \left( \frac{2\pi ux}{M} \right) \right] \text{ for } u = 0, 1, \ldots, M-1. \]

For each term of the Fourier transform \( \{F(u)\} \) is made up of all values of the function \( f(x) \).

The discrete Fourier transform always exist!
The Fourier Transform

Discrete 1D case

Express $F(u)$ in polar coordinates

$$F(u) = |F(u)| e^{-j\theta(u)}$$

where the magnitude (spectrum) of the Fourier spectrum is

$$|F(u)| = \left[ R^2(u) + I^2(u) \right]^{1/2}$$

and the phase angle (phase spectrum) is

$$\Theta(u) = \tan^{-1} \left( \frac{I(u)}{R(u)} \right).$$

The power spectrum (spectral density) is

$$P(u) = |F(u)|^2 = R(u)^2 + I(u)^2.$$
The Fourier Transform

The discrete sample spacing is related by

$$\Delta u = \frac{1}{M \Delta x}.$$ 

**Note!**

- Each term in FT (each $u$ or $(u, v)$) is composed of the sum of all values of $f(x)$ or $f(x y)$. 

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The Fourier Transform

Properties of the FT

- **Separable**: The 2D Fourier transform can be performed by two 1D transforms

\[ f(x, y) \rightarrowrow F(x, v) \rightarrowcolumn F(u, v). \]

- **Linear**: 

\[ \mathcal{F} (af(x) + bg(x)) = aF(u) + bG(u). \]

- **Translation**: 

\[ \mathcal{F} (f(x - x_0)) = F(u) e^{-i2\pi ux_0/M} \quad \mathcal{F} \left( f(x) e^{i2\pi xu_0/M} \right) = F(u - u_0). \]

- **Rotation**: Polar coordinates. *The angles are the same in spatial and frequency domain.*

\[ x = r \cos(\theta) \quad u = w \cos(\phi) \]
\[ y = r \sin(\theta) \quad v = w \sin(\phi) \]
\[ \mathcal{F} (f(r, \theta + \theta_0)) = F(w, \phi + \theta_0) \]
The Fourier Transform
GW example 4.1

A discrete function of $M$ points, and its Fourier spectrum.

A discrete function with twice the number of nonzero points, and its Fourier spectrum.
The Fourier transform

GW example 4.1: Implemented in MATLAB

```matlab
M = 1024;
A = 1;
K = 8;
f = zeros(M, 1);
f(1:K) = ones(K, 1);
R = zeros(M, 1);
I = zeros(M, 1);
for u = 0:M-1
    for x = 0:M-1
        R(u+1) = R(u+1)+((-1)^x*f(x+1)*cos(2*pi*u*x/M));
        I(u+1) = I(u+1)+((-1)^x*f(x+1)*sin(2*pi*u*x/M));
    end
    F(u+1) = 1/M*sqrt(R(u+1)*R(u+1)+I(u+1)*I(u+1));
end
```
The Fourier Transform

1D Discrete Fourier Transform

1D image \((x = 0..1024)\)  

1D Fourier transform \((u = 0..1024)\)
The Fourier Transform
Inverse 1D DFT

```matlab
invF = zeros(M, 1);
for x = 0:M-1
    for u = 0:M-1
        invF(x+1) = invF(x+1) + (R(u+1)*cos(2*pi*u*x/M) + I(u+1)*sin(2*pi*u*x/M));
    end
    invF(x+1) = (-1)^x * (1/M) * invF(x+1);
end
```
The Fourier Transform
GW example 4.2

Left  Image of a $20 \times 40$ white rectangle on a black background of size $512 \times 512$ pixels.

Right Centered Fourier spectrum shown after application of the log transformation $D(x, y) = c \log (1 + |F(u, v)|)$. 

The Fourier Transform

Original.  

Low frequencies.  

High frequencies.
The Fourier Transform
The Fourier Transform

[Images of a star and a grid pattern, possibly related to frequency domain analysis]
The Fourier Transform

Rotation

The Fourier Transform can be used in combination with immunostaining and segmentation, which, in combination with immunostaining and segmentation, can be used to detect antigens that can be observed in tissue sections. Visualization of multicolor immunostaining is necessary, both for quantitative analysis and relationships of functional signaling, meaning repeated application of repeated markers, greatly increases the number of antigens that can be visualized and analyzed.

\[ f(x, y) \]

\[ F(f(x, y)) \]
The Fourier Transform
Filtering in the frequency domain

Basic idea:

1. Fourier transform an image.
2. Manipulate the transform by suppressing certain parts.
3. Inverse transform and get a new “better” image.

Most common is to use a filter which affects the real and the imaginary part of the transform equally, i.e., the phase is not changed (“zero-phase shift filters”).
The Fourier Transform
Convolution (swedish “faltning”)

- Convolution of two continuous 1D functions $f(x)$ and $h(x)$ is defined as
  \[ f(x) \ast h(x) = \int_{-\infty}^{+\infty} f(\alpha)h(x - \alpha) \, d\alpha. \]

- In 2D we get
  \[ f(x, y) \ast h(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha, \beta)h(x - \alpha, y - \beta) \, d\alpha \, d\beta. \]

- Convolution of two discrete functions $f(x, y)$ and $h(x, y)$
  \[ f(x, y) \ast h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x - m, y - n) \]
The Fourier Transform
Convolution

- The impulse function is defined as
  \[ \int_{-\infty}^{+\infty} f(x) \delta(x - x_0) \, dx = f(x_0). \]

- Convolution by the impulse function
  \[ f(x) * \delta(x - x_0) \]
  results in a “copy” of \( f(x) \) at the location of the impulse.
The Fourier Transform

The Convolution theorem

- States that
  \[ f(x, y) \ast h(x, y) \Leftrightarrow F(u, v) \cdot H(u, v) \]
  and
  \[ f(x, y) \cdot h(x, y) \Leftrightarrow F(u, v) \ast H(u, v) , \]
  i.e., convolution in the spatial domain corresponds to multiplication in the frequency domain, and vice versa.

- The convolution theorem makes up the fundamental link between image processing in the spatial domain and the frequency domain.
The Fourier Transform
Low pass filters

- We “remove” high frequencies. This corresponds to a smoothing filter in the spatial domain.
- The simplest case is to use an ideal lowpass filter (ILPF) where $H(u, v)$ is set to 0 outside a given frequency.

$$H(u, v) = \begin{cases} 
1 & \iff D(u, v) \leq D_0 \\
0 & \iff D(u, v) > D_0 
\end{cases}$$

- Problem: ILPF results in “rings” in the image. These artefacts are avoided by not using a sharp cut-off frequency. A smooth cut-off gives a nicer result (e.g., Butterworth, Gauss. etc.).
The Fourier Transform

Lowpass filtering

Original

Low frequencies

\[ G = HF \]
The Fourier Transform

Highpass filters

- We “remove” low frequencies. This corresponds to an edge enhancing filter in spatial domain.

- The simplest case is to use an ideal highpass filter (IHPF) where $H(u, v)$ is set to 0 inside a given frequency.

\[
H(u, v) = \begin{cases} 
0 & \iff D(u, v) \leq D_0 \\
1 & \iff D(u, v) > D_0 
\end{cases}
\]

- Problem: IHPF will just like LPF result in “rings” in the image. A smooth cut-off gives a nicer result (e.g., Butterworth, Gauss, etc.).
The Fourier Transform

Highpass filtering

Original

High frequencies
Other filters

**Bandpass**: Allow frequencies in a band between two frequencies $D_0$ and $D_1$.

**Bandstop**: Stops frequencies in a band between two frequencies $D_0$ and $D_1$.

**Other**: Filters which allow different frequencies in the $u$, and $v$ direction.
The Fourier Transform

Conclusions

- The Fourier transform and the convolution theorem make up the fundamentals of image processing in the frequency domain.
- Filtering in the frequency domain gives the same results as linear filtering in the spatial domain.
- Why filter in the frequency domain?
  - Save time.
  - Opens the possibilities of image processing which is simply not possible in the spatial domain (see lecture on FFT).
The Fourier Transform

Example

original $f(x,y)$

$F\{f(x,y)\} = F(u,v)$

$G(u,v) = H(u,v)F(u,v)$

$F^{-1}\{G(u,v)\} = f'(x,y)$

"ring" effects
The Fourier Transform
More examples

\[ f(x, y) \]

\[ F(f(x, y)) \]
The Fourier Transform

More examples
Reading Instructions

Chapters next lecture

- Chapter 10.1 – 10.2.5 and 10.3 – 10.5 in Gonzales-Woods.