Today’s lecture

• SE, morphological transformations
  • Binary MM
  • Gray-level MM
  • Applications
  • Geodesic transformations

Mathematical Morphology
Sonka 13.1-13.6
Ida-Maria Sintorn
ida@cb.uu.se

Morphology-form and structure

Mathematical framework used for:

• pre-processing
  • noise filtering, shape simplification, ...
• enhancing object structure, describing shape
  • skeletonization, convex hull...
• segmentation
• quantitative description
  • area, perimeter, ...

structuring element (SE)

• small set, B, to probe the image under study
• for each SE, define origo & pixels in SE
• shape and size must be adapted to geometric properties for the objects

Morphological Transformation

• \( \psi \) is given by the relation of the image (point set X) and the SE (point set B).
• in parallel for each pixel (pixel under SE origo) in binary image:
  – check if SE is “satisfied”
  – output pixel is set to 0 or 1 depending on used operation

Five binary morphological transforms

\( \oplus \) Erosion, shrinking
\( \ominus \) dilation, growing
\( \circ \) opening, erosion + dilation
\( \bullet \) closing, dilation + erosion
\( \otimes \) Hit-or-Miss transform
Erosion (shrinking)
For which points does the structuring element fit the set?
erosion of a set X by structuring element B, $\varepsilon_B(X)$:
all $x$ in $X$ such that $B$ is in $X$ when origin of $B=x$
\[
x \ominus B = \varepsilon_B(X) = \{x \mid B_x \subseteq X\}
\]

Dilation (growing)
For which points does the structuring element hit the set?
dilation of a set X by structuring element B, $\delta_B(X)$:
all $x$ such that the reflection of $B$ hits $X$ when origin of $B=x$
\[
X \oplus B = \delta_B(X) = \{x \mid (B)_x \cap X \neq \emptyset\}
\]

Combining erosion and dilation
WANTED:
remove structures / fill holes without affecting remaining parts
SOLUTION:
combine erosion and dilation (using same SE)

Opening
Closing

Opening
erosion followed by dilation
eliminates protrusions, breaks necks, smoothes contours

Closing
dilation followed by erosion, denoted •
Smoothes contours, fuses breaks, eliminates holes and gaps

Opening: roll ball (=SE) inside object
see B as a "rolling ball"
boundary of $A \oplus B$ = points in B that reaches closest
to A boundary when B is rolled inside A

Closing: roll ball (=SE) outside object
boundary of $A \ominus B$ = points in B that reaches closest
to A boundary when B is rolled outside A
Exercise

- Sketch the result of A first eroded by B1 and then dilated by B2

\[ A \ominus B = (A \ominus B_1) \cap (A^c \ominus B_2) \]

composite SE: object part \( B_1 \) and background part \( B_2 \)
does \( B_1 \) fits the object while, simultaneously, \( B_2 \) misses the object, i.e., \( \text{fits the background} \)

hit-or-miss transformation \( \ominus \) (HMT)
find location of one shape among a set of shapes
"template matching"

erosion and dilation are dual with respect to complementation and reflection
\[ (A \ominus B)^c = A^c \oplus \hat{B} \]

Gray-level images and SEs

Same SE (flat), gray level description
\( f(0,0) = 0 \)
\( f(-1,0) = 0 \)
\( f(1,0) = 0 \)
\( f(0,-1) = 0 \)
\( f(0,1) = 0 \)
\( \text{Domain(f): } (0,0),(-1,0),(1,0),(0,-1),(0,1) \)
Think topographically

Gray-level SE (not flat!)
- \( f(0,0) = 1 \)
- \( f(-1,0) = 0 \)
- \( f(1,0) = 0 \)
- \( f(0,-1) = 0 \)
- \( f(0,1) = 0 \)

Domain(f) = \{(0,0),(1,0),(0,-1),(0,1)\}

Top surface & umbra

Umbra homeomorphism theorem

Umbra operation is a homeomorphism from grayscale morphology to binary morphology

\[ f \oplus b = T \{ U[f] \oplus U[b] \} \]

Gray-scale umbra erosion

Gray scale erosion of two functions as (binary) erosion of umbras

\[ \varepsilon_{U(B)}(U(f)) \]

\[ \varepsilon_B(f) \]
Gray-scale umbra erosion

\[
\varepsilon_B(f)(x) = \min_{b \in \mathbb{Z}^2_B} \{ f(x + b) - B(b) \}
\]

Gray-scale Morphological erosion

\[
\varepsilon_B(f)(x) = \min_{b \in \mathbb{Z}^2_B} \{ f(x + b) - B(b) \}
\]

Gray scale erosion

\[
\varepsilon_B(f)(x) = \min_{b \in \mathbb{Z}^2_B} \{ f(x + b) - B(b) \}
\]

\( B(0)=0 \)
\( B(1)=1 \)
\( B(2)=0 \)
Gray scale erosion

\[
\delta_B(f) = \min_{b \in \mathbb{F}} \{ f(x+b) - B(b) \}
\]

\[
\delta_B(f)(2) = \min(6,0,4,1,6,0) = 3
\]

Example, gray-scale erosion

flat SE, square 3x3

- b with positive elements \(\rightarrow\) darker output
- bright details are reduced
- If flat SE, erosion is min of \(f-b\)

Gray scale dilation

\[
\delta_U(f) = \delta_B(U(f))
\]

Gray scale dilation

\[
\delta_U(f) = \delta_B(U(f))
\]
Gray scale dilation

\[ \delta_B(f)(x) = \max_{b \in D_B} \{ f(x-b) + B(b) \} \]

\[ \delta_B(f)(0) = \max(2+0,0+1,0+0) = 2 \]

\[ \delta_B(f)(1) = \max(4+0,2+1,0+0) = 4 \]
Gray scale dilation

\[ \delta_B(f) (x) = \max \{ f(x - b) + B(b) \} \]

\[ \delta_B(f) (2) = \max (6 + 0, 4 + 1, 2 + 0) = 6 \]

Example, gray-scale dilation

- flat SE, square 3x3
  - SE with positive elements \( \rightarrow \) brighter output
  - dark details are reduced or eliminated
  - if flat SE, dilation is max of \( f + b \)

Morphological opening

\[ \gamma_B(f) = \delta_B [ \varepsilon_B(f) ] \]

Example, gray-scale opening, flat SE, square 3x3

- remove small bright details
- leave overall gray-levels
- leave larger bright features

Morphological closing

\[ \phi_B(f) = \varepsilon_B [ \delta_B(f) ] \]
Example, gray-scale closing, flat SE, square 3x3

- remove dark details
- Leave overall gray-levels
- leave bright features

Morphological gradient

\[ (X \ominus B) \setminus (X \oplus B) \]

Dilated image – eroded image

Morphological smoothing

removal or attenuation of bright & dark artifacts/noise

\[ (X \circ B) \bullet B \]

Opening followed by closing

Top hat transformation

\[ X \setminus (X \circ B) \]

Highlight/segment features of certain size & shape, correct for uneven background

Use SE slightly larger than objects you want to highlight

Which operation?

SE circle, radius=4
Erosion, dilation, opening, closing

geometrical 1D interpretation

Gray-level opening and closing
(from GW)

opening

closing
Another top hat example

Gray scale hit-or-miss

$\text{Gray scale hit-or-miss}$

Problem:
Find vessels in the 2D mip image.
(The 3D image is acquired by MR.)

Result with

Granulometry
"Measurement of grain sizes of sedimentary rock"

- Measuring particle size distribution indirectly
- Shape information without
  - segmentation
  - separated particles
- Apply morphological openings of increasing size
- Compute the sum of all pixel values in the opening $\Rightarrow$ surface area of the image
Example, coin image

The peaks correspond to the size of the elements!

```
sumpixels=zeros(1,maxsize+1);
for k=0:maxsize
    se=strel('disk',k);
    fo=imopen(f,se);
    sumpixels(k+1)=sum(fo(:));
end
```

Original image, openings of discs with radii 19, 22, 25, 29

Geodesic transformations

**Geodesic dilation**

Input: marker image \(f\) and mask image \(g\).
- Dilate the marker image \(f\) with the unit ball.
- Output the minimum value of the dilation of \(f\) and the mask image \(g\)

\[
\delta^{(1)}_g(f) = \delta(f) \land g
\]

Geodesic erosion

Input: marker image \(f\) and mask image \(g\).
- Erode the marker image \(f\) with the unit ball.
- Output the maximum value of the erosion of \(f\) and the mask image \(g\)

\[
\varepsilon^{(1)}_g(f) = \varepsilon(f) \lor g
\]

Geodesic transformations

**Geodesic dilation example**

- Marker
- Mask
- Marker dilated
- Dilated mask
- Result

Geodesic transformations

**Geodesic erosion example**

Morphological reconstruction

- \(X\) is set of connected components \(X_1, \ldots, X_n\). \(Y\) is markers in \(X\).
- Reconstruction by dilation: Geodesic dilations until stability.
- Reconstruction by erosion: Geodesic erosions until stability.
Morphological reconstruction

- Reconstruction by dilation: Geodesic dilations until stability.
  \[ \delta_g^{(n)}(f) = \underbrace{\delta_g^{(1)}(\delta_g^{(1)}(\ldots \delta_g^{(1)}(f)))}_{n \text{ times}} \]

- Reconstruction by erosion: Geodesic erosions until stability.
  \[ \varepsilon_g^{(n)}(f) = \underbrace{\varepsilon_g^{(1)}(\varepsilon_g^{(1)}(\ldots \varepsilon_g^{(1)}(f)))}_{n \text{ times}} \]

Reconstruction by erosion: Minima imposition

First iteration: Erode marker image with elementary SE.

Second iteration: Erode result from first iteration with elementary SE.

When stability is reached:

All local minima except for the marked minimum are removed!

This can be used for seeded watershed!
Application - Seeded watershed by Minima imposition

Seeded watershed by Minima imposition

Application - Image compositing

Two images should be merged.
Decide where the “seam” should be.

Image compositing

Compute gradient.
Do seeded watershed with minima imposition.
(Seeds on the border of the image.)

Image compositing

Result