Mathematical Morphology

Sonka 13.1-13.6
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Today’s lecture

• SE, morphological transformations
• Binary MM
• Gray-level MM
• Applications
• Geodesic transformations

Morphology-form and structure

mathematical framework used for:

• pre-processing
  • noise filtering, shape simplification, ...
  • enhancing object structure, describing shape
  • skeletonization, convex hull...
• segmentation
• quantitative description
  • area, perimeter, ...

structuring element (SE)

• small set, B, to probe the image under study
• for each SE, define origo & pixels in SE
• shape and size must be adapted to geometric properties for the objects

Morphological Transformation

• $\psi$ is given by the relation of the image (point set X) and the SE (point set B).
• in parallel for each pixel (pixel under SE origo) in binary image:
  – check if SE is “satisfied”
  – output pixel is set to 0 or 1 depending on used operation

Five binary morphological transforms

- $\varepsilon$ Erosion, shrinking
- $\delta$ dilation, growing
- $\gamma$ opening, erosion + dilation
- $\phi$ closing, dilation + erosion
- $\otimes$ Hit-or-Miss transform
**Erosion (shrinking)**

For which points does the structuring element fit the set?

Erosion of a set $X$ by structuring element $B$, $\varepsilon_B(X)$:

all $x$ in $X$ such that $B$ is in $X$ when origin of $B=x$

$$x \ominus B = \varepsilon_B(X) = \{ x \mid B \subseteq X \}$$

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**Dilation (growing)**

For which points does the structuring element hit the set?

Dilation of a set $X$ by structuring element $B$, $\delta_B(X)$:

all $x$ such that the reflection of $B$ hits $X$ when origin of $B=x$

$$X \oplus B = \delta_B(X) = \{ x \mid (\hat{B})_x \cap X \neq 0 \}$$

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**Duality**

Erosion and dilation are dual with respect to complementation and reflection

$$(A \ominus B)^C = A^C \oplus \hat{B}$$

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**Combining erosion and dilation**

**WANTED:**
remove structures / fill holes without affecting remaining parts

**SOLUTION:**
combine erosion and dilation (using same SE)

- **Opening**
- **Closing**

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**Opening**

Erosion followed by dilation eliminates protrusions, breaks necks, smooths contours

$$A \circ B = (A \ominus B) \oplus B$$

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**Closing**

dilation followed by erosion, denoted •
Smoothes contours, fuses breaks, eliminates holes and gaps

$$A \bullet B = (A \oplus B) \ominus B$$
opening: roll ball (=SE) inside object
see B as a “rolling ball”
boundary of $A \circ B$ = points in B that reaches closest
to A boundary when B is rolled inside A

closing: roll ball (=SE) outside object
boundary of $A \circ B$ = points in B that reaches closest
to A boundary when B is rolled outside A

Exercise

• Sketch the result of A first eroded by B1
  and then dilated by B2

hit-or-miss transformation (⊕,HMT)
find location of one shape among a set of shapes
“template matching”

$A \otimes B = (A \Theta B_1) \cap (A^C \Theta B_2)$

composite SE: object part ($B_1$) and background part ($B_2$)
does $B_1$ fit the object while, simultaneously, $B_2$ misses the object, i.e., fits the background?

hit-or-miss transformation (⊕,HMT)

Gray-level images and SEs

Same SE (flat), gray level description
$f(0,0)=0$
$f(-1,0)=0$
$f(1,0)=0$
$f(0,-1)=0$
$f(0,1)=0$
$\text{Dom}(f) = \{(0,0),(-1,0),(1,0),(0,-1),(0,1)\}$
**Think topographically**

Gray-level SE (not flat!)
- \( f(0,0) = 1 \)
- \( f(-1,0) = 0 \)
- \( f(1,0) = 0 \)
- \( f(0,-1) = 0 \)
- \( f(0,1) = 0 \)

Domain(\( f \)) = \{ (0,0), (-1,0), (1,0), (0,-1), (0,1) \}

**Top surface & umbra**

1D function \( f \) (top surface, \( T \))
- Its umbra \( U[f] \)

**Umbra homeomorphism theorem**

Umbra operation is a homeomorphism from grayscale morphology to binary morphology

\[
\forall \, b \in T \{ U[f] \oplus U[b] \}
\]

**Gray-scale umbra erosion**

Gray scale erosion of two functions as (binary) erosion of umbras

\[
\varepsilon_{U[B]}(U(f)) \quad \varepsilon_{B}(f)
\]

**Gray-scale umbra erosion**

Gray scale erosion of two functions as (binary) erosion of umbras

\[
\varepsilon_{U[B]}(U(f)) \quad \varepsilon_{B}(f)
\]
Gray-scale umbra erosion

\[ \{e_B(f)\}(x) = \min_{b \in \{0,1,2\}} \{ f(x + b) - B(b) \} \]

Gray-scale Morphological erosion

\[ \{e_B(f)\}(x) = \min_{b \in \{0,1,2\}} \{ f(x + b) - B(b) \} \]

Gray scale erosion

\[ \{e_B(f)\}(x) = \min_{b \in \{0,1,2\}} \{ f(x + b) - B(b) \} \]

\[ \{e_B(f)\}(0) = \min(2-0,4-1,6-0) = 2 \]

\[ \{e_B(f)\}(1) = \min(4-0,6-1,4-0) = 4 \]
Gray scale erosion

\[ \varepsilon_B(f)(x) = \min_{b \in \text{domain}(B)} \{ f(x + b) - B(b) \} \]

\[ \varepsilon_B(f)(2) = \min(6-0,4-1,6-0) = 3 \]

Example, gray-scale erosion
flat SE, square 3x3

- b with positive elements \( \rightarrow \) darker output
- bright details are reduced
- If flat SE, erosion is min of f-b

Gray scale dilation of two functions as (binary) dilation of umbras

\[ \delta_{U(B)}(U(f)) \]
\[ \delta_B(f) \]
Gray scale dilation

\[ [\delta_B(f)](x) = \max_{b \in \mathcal{D}_B} \{f(x - b) + B(b)\} \]

\[ [\delta_B(f)](0) = \max(2+0,0+1,0+0) = 2 \]

\[ [\delta_B(f)](1) = \max(4+0,2+1,0+0) = 4 \]
Gray scale dilation

$$\delta_B(f)(x) = \max_{b \in \mathbb{B}_B} \{ f(x - b) + B(b) \}$$

$$\delta_B(f)(2) = \max(6+0, 4+1, 2+0) = 6$$

Example, gray-scale dilation
flat SE, square 3x3

- SE with positive elements → brighter output
- dark details are reduced or eliminated
- If flat SE, dilation is max of f+b

Morphological opening

$$\gamma_B(f) = \delta_B[\varepsilon_B(f)]$$

Example, gray-scale opening, flat SE, square 3x3

- remove small bright details
- leave overall gray-levels
- leave larger bright features

Morphological closing

$$\phi_B(f) = \varepsilon_B[\delta_B(f)]$$
Example, gray-scale closing, flat SE, square 3x3

- remove dark details
- Leave overall gray-levels
- leave bright features

Which operation (erosion, dilation, opening, closing) is applied to the image above? SE circle, radius=4

Morphological gradient

\[(X \oplus B \setminus X \ominus B)\]
dilated image – eroded image

Other ways?

Morphological smoothing

removal or attenuation of bright & dark artifacts/noise

\[(X \ast B) \ast B\]
Opening followed by closing

Top hat transformation

Highlight/segment features of certain size & shape, correct for uneven background
Use SE slightly larger than objects you want to highlight

\[X \setminus (X \circ B)\]
original image – opened image
Another top hat example

Gray scale hit-or-miss
\[ \text{UHMT}_B(f)(x) = \begin{cases} |B_{FG}(f)(x)| - |B_{BG}(f)(x)|, & \text{if } |B_{FG}(f)(x)| < |B_{BG}(f)(x)| \\ 0, & \text{otherwise} \end{cases} \]

where $B_{FG}$ is the SE for the object (foreground) and $B_{BG}$ is the SE for the background. Basically an erosion minus a dilation.

Gray scale hit-or-miss

Problem:
Find vessels in the 2D miP image.
(The 3D image is acquired by MR.)

Light gray: $B_{FG}$
Dark gray: $B_{BG}$

Result with

Sum of results:

Gray scale hit-or-miss

Problem:
Find vessels in the 3D MR image.

SEs: Rotations of this SE.

Granulometry
"Measurement of grain sizes of sedimentary rock"

- Measuring particle size distribution indirectly
- Shape information without
  - segmentation
  - separated particles
- Apply morphological openings of increasing size
- Compute the sum of all pixel values in the opening \( \Rightarrow \) surface area of the image
Example, coin image

The peaks correspond to the size of the elements!

Original image, openings of discs with radii 19, 22, 25, 29

Geodesic transformations

**Geodesic dilation**

Input: marker image $f$ and mask image $g$.
- Dilate the marker image $f$ with the unit ball.
- Output the minimum value of the dilation of $f$ and the mask image $g$

$$\delta_g^{(1)}(f) = \delta(f) \wedge g$$

Geodesic transformations

**Geodesic erosion**

Input: marker image $f$ and mask image $g$.
- Erode the marker image $f$ with the unit ball.
- Output the maximum value of the erosion of $f$ and the mask image $g$

$$\varepsilon_g^{(1)}(f) = \varepsilon(f) \vee g$$

Geodesic transformations

**Geodesic dilation example**

Marker                  mask                  marker dilated         dilated and mask           result

Geodesic transformations

**Morphological reconstruction**

- X is set of connected components $X_1,...,X_n$. Y is markers in X.
- Reconstruction by dilation: Geodesic dilations until stability.
- Reconstruction by erosion: Geodesic erosions until stability.
Morphological reconstruction

- Reconstruction by dilation: Geodesic dilations until stability.
  \[ \delta^{(n)}_g(f) = \delta^{(1)}_g(\delta^{(1)}_g(\ldots \delta^{(1)}_g(f))) \]
  \( n \) times

- Reconstruction by erosion: Geodesic erosions until stability.
  \[ \varepsilon^{(n)}_g(f) = \varepsilon^{(1)}_g(\varepsilon^{(1)}_g(\ldots \varepsilon^{(1)}_g(f))) \]
  \( n \) times

Reconstruction by erosion: Minima imposition

First iteration: Erode marker image with elementary SE.
Pointwise max of \( f \) and \( g \)

Second iteration: Erode result from first iteration with elementary SE.
Pointwise max of \( f \) and \( g \)

Reconstruction by erosion: Minima imposition

When stability is reached:
All local minima except for the marked minimum are removed!
This can be used for seeded watershed!
Application - Seeded watershed by Minima imposition

Input image

Watershed on edge image.

(Oversegmentation?)

Edge image by morphological gradient 

\[ \delta(f), \varepsilon(f) \]

Application - Image compositing

Two images should be merged.

Decide where the “seam” should be.

Image compositing

Result

Seeded watershed by Minima imposition

Seeds

Watershed on the minima imposition

Minima imposition using the seeds as markers

Image compositing

Compute gradient.

Do seeded watershed with minima imposition.

(Seeds on the border of the image.)

Image compositing

Result