Digital topology
Skeletonization
Medial axis transform

Often described as being the “locus of local maxima” on a distance map.

Augmented by radial function, the *quench* function.

Blum 1967
Medial axis representation

Compact representation of objects.

Applications:

- Object description
  - The object should be fully described by the representation

- Object recognition

- Navigation
  - Only the most important features are needed

- Animation

- ...

- ...

Topology

- Description invariant under "rubber sheet" transformation

Terminology:
- homotopy
- homeomorphism
- topologically equivalent

Homotopy equivalent, but not homeomorphic
Medial axis representation in digital images

This can be done in different ways, for example:

- Centers of maximal balls (CMBs)
- Homotopic thinning
- Homotopic thinning keeping the CMBs
- Template matching

Different approaches give different properties of the medial axis.
Medial axis representation in digital images

Centers of maximal balls
Medial axis representation in digital images

- **Maximal ball** – ball in the object that is not covered by any other ball in the object.
- **CMB** – its center.

Depends on the distance function!

The **Quench function** associates the radius to each CMB.

Compare with the continuous case.
Pixels as centers of balls

Distance label of pixel $p$ can be interpreted as radius of a ball $B(p,d(p))$, centered on $p$

$B(p,d(p))$ is fully enclosed in the object

chessboard distance
Centers of maximal balls

If not completely covered by any other disc

Note: Not all CMBs needed for reconstruction
Centers of maximal balls (CMB)

- Appear as local maxima in DT for weighted distances(!)
- Union of all discs corresponding to CMBs = object

- cityblock
- chessboard
- $<3,4>$
- Euclidean
Centers of maximal balls

A pixel is a center of maximal ball if it is a local maximum in the DT. (note! take local distance into account)

for pixel in \(<a,b>\) WDT labeled \(p\):
  edge neighbors < \(p+a\)
  vertex neighbors < \(p+b\)
  for city-block:
    edge neighbors have lower or equal label
  for chessboard:
    neighbors have lower or equal label
Centers of maximal balls

Original image

Sets of CMBs with different distance functions

city-block                       chessboard                 <3,4>-weighted
cityblock is used for this example
Complete description by CMBs

Object can be represented by its CMBs as it is the union of the maximal balls

Reverse distance transformation can be used to recover the object

object

object = grey
CMBs = black

reverse DT
Reverse DT from CMBs

max-operation

<3,4> weighted

Propagate from CMBs (in bold)

after forward scan

backward scan
Centers of maximal balls for Euclidean DT

- Not enough to check distance values of neighbors
- Maximal ball: not covered by any other single ball

Remember: the 3x3 neighborhood does not hold enough information about the Euclidean distance

- Simple local comparisons not enough: use look-up tables
Medial axis representation in digital images

Homotopic thinning using simple points
Simple pixels

Pixels that can be removed without altering topology:
• the number of object components and
• the number of background components
are the same before and after removal
Simple pixels by local neighborhood operations

Decision on whether a pixel is simple or not can be taken based on local neighborhood configuration. For 8-connected object and 4-connected background:

\( N^8(v) \) number of object components in an 8-neighborhood of \( v \)

\( \overline{N}^8(v) \) number of background components in an 8-neighborhood of \( v \), edge connected to \( v \)

\( v \) is simple if

\[ N^8(v) = 1 \]

\[ \overline{N}^8(v) = 1 \]
Homotopic thinning

- Remove border after border if simple pixel
- Number of iterations is dependent on object thickness

Repeat until stability:

- Find border pixels
- Remove border pixels if simple

OR use distance transform to define borders!
Homotopic thinning

Original image

Result after homotopic thinning (removing only simple points).
Medial axis representation in digital images

Homotopic thinning keeping the CMBs
Homotopic thinning keeping the CMBs

Keep CMBs and remove simple points sequentially

- Compute distance transform
- Remove border after border if
  - not a CMB
  - simple pixel
- Number of iterations is dependent on object thickness
Homotopic thinning keeping the CMBs

Original image

Homotopic thinning keeping the CMBs with different distance functions

city-block                       chessboard                 <3,4>-weighted
object

CMBs

<3,4>-weighted

result
Homotopic thinning keeping the CMBs with different DTs

city block  chessboard  $\langle 3,4 \rangle$  Euclidean

Different aspects:
• shape preservation
• compression
• stability under rotation
Medial axis representation in digital images

Homotopic thinning by template matching
Thin using morphology

- Sequential thinning by a sequence of structuring elements (SE, "masks")
  - Application of hit-or-miss
  - Identify border pixels (use DT)
  - Remove pixels satisfying one SE
  - Composite SEs: object, background, don’t care

\[
\begin{align*}
L_1 &= \begin{bmatrix}
0 & 0 & 0 \\
\ast & 1 & \ast \\
1 & 1 & 1
\end{bmatrix} & L_2 &= \begin{bmatrix}
\ast & 0 & 0 \\
1 & 1 & 0 \\
\ast & 1 & \ast
\end{bmatrix} & \text{...}
\end{align*}
\]

L from Golay alphabet gives homotopic thinning
Thinning by template matching

Original image

Thinning by template matching using the templates on the previous slide.
Skeletal properties

- In an image with object O and background B, the skeleton S is categorized by the following properties
  - S is topologically equivalent to O
  - S is centered within O
  - S is unit-wide
  - O is recovered by reversing S

Sometimes skeleton is defined as a transformation having all these properties.
Skeletal properties

- S is topologically equivalent to O  no
- S is centered within O      yes
- S is unit-wide            no
- O is recovered by reversing S  yes

The set of CMBs
Skeletal properties

- S is topologically equivalent to O  yes
- S is centered within O  yes
- S is unit-wide  yes
- O is recovered by reversing S  no
Skeletal properties

Homotopic thinning keeping the CMBs

- S is topologically equivalent to O  yes
- S is centered within O          yes
- S is unit-wide                no
- O is recovered by reversing S  yes
Skeletal properties

- S is topologically equivalent to O  yes
- S is centered within O  yes
- S is unit-wide  yes
- O is recovered by reversing S  no

Thinning by template matching
Similar methods as in 2D apply to 3D. We need to define

- Homotopic transformations,
- Simple points, and
- CMBs in 3D.
Skelettons in 3D

Basic notions

**concavity**
- dent on the object

**tunnel**
- background passing through the object

**cavity**
- background component enclosed in the object
Homotopic transformation

Here, a transformation is homotopic (topology preserving) if it can be written as a sequence of adding/removing simple points.

In 2D, the number of components and holes remain unchanged under the transformation.

In 3D, the number of object components, the number of cavities and the number of tunnels remain unchanged.
A point is **simple** iff its removal does not alter the topology of the object and background.

Can be detected in a similar way as for 2D images.
Balls generated by different metrics

- $D^6$: Unit weight to face neighbors
- $D^{26}$: Unit weight to face, edge, and vertex neighbors
- $\langle 3,4,5 \rangle$: Weight 3,4,5 to face, edge, and vertex neighbors, respectively
- Euclidean
Centers of maximal balls

As in 2D, a voxel is a center of maximal ball if it is a local maximum in the DT for weighted distances.

*(note! take local distance into account)*

for voxel in \(<a,b,c>\) WDT labeled \(v\):

- face neighbors < \(v+a\)
- edge neighbors < \(v+b\)
- vertex neighbors < \(v+c\)

for \(D^6\):
- face neighbors have lower or equal label

for \(D^{26}\):
- neighbors have lower or equal label
Skeletonization in 3D

Surface skeleton is obtained by keeping CMBs and removing simple points sequentially.

- 3D object $\rightarrow$ 2D surface skeleton $\rightarrow$ 1D curve skeleton
- reversibility can only be guaranteed from surface skeleton