Analysis of digital shapes: Can uncertainty help us to measure more precisely?

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1 Analysis of digital shapes: Crisp approach

2 Introducing uncertainty: Fuzzy approach

3 A special case: Coverage model

4 Concluding remarks
Image analysis and Image processing

• The task of Image Analysis is to extract relevant information from images.
• Numerical descriptors, such as area, perimeter, and moments of objects are often of interest, for the tasks of shape analysis, classification, etc.
Image analysis and Image processing

- The task of **Image Analysis** is to **extract relevant information from images**.
- **Numerical descriptors**, such as area, perimeter, and moments of objects are often of interest, for the tasks of shape analysis, classification, etc.

### The standard image analysis task (and its solution)

1. Sample preparation and Imaging
2. Pre-processing (optional)
3. Segmentation
   - Discrete representation of a continuous object of interest
4. Feature extraction
   - Information about continuous object, based on discrete data
5. Classification, statistical evaluation, ...
Important step - image segmentation

For every pixel in the image, decide if it belongs to the object (foreground) or to the background.

\[ r = 3.60 \]
\[(c_x, c_y) = (5.68, 6.41)\]

\[ r = 3.50 \]
\[(c_x, c_y) = (5.62, 6.44)\]

Many criteria can be used. Here, pixels with centres covered by the continuous object are considered foreground. Others belong to background.

The decision is crisp - the pixel is in, or out.
Feature estimation

• We derive conclusions about objects of interest based on a number of their relevant features.
• We are interested in **continuous objects**, while having available only their **discrete representations**.
• We wish to obtain feature values as similar to the correct ones as possible - **accuracy**
• We wish that repeated measurements provide as similar results as possible - **precision**.
• Due to loss of information, measurements are in general neither accurate, nor precise.
• We need to
  • define estimators;
  • control the estimation error.
Crisp segmentation and feature estimation

Not only that different objects may have same discrete representation, but also one object may have different discrete representations, depending on its position in the discretization grid.

Crisp discrete object representations, especially at low spatial resolutions, put strong limitations to precision of feature estimation:
Area estimation

How to estimate:

Rather intuitive and very simple - we count the object pixels.

How big is the error?

- This appears to be a very difficult problem.
- Gauss tried to estimate the upper bound for the error for circles. This is still an open problem...
- It is clear that the error decreases with an increase of a size of a disk or, equivalently, with a decrease of the size of a pixel.

So, for high enough resolution, we will obtain precise enough area estimate by counting the foreground pixels (and scaling appropriately).
Area estimation error

**Theorem**

The area of a closed bounded set $S$, digitized in a grid with resolution $r$ (the number of grid points per unit), can be estimated by

$$
\text{Area}(S) = \frac{1}{r^2} \tilde{\text{Area}}(rS) + O \left( \frac{1}{r} \right).
$$

Here $rS$ denote a scaling of the continuous set $S$ about the origin by the factor $r$. Scaling of the object can be used instead of changing resolution of a grid.

$\tilde{\text{Area}}(rS)$ is the estimate of the area of $rS$; it is equal to the number of pixels in discrete representation of $rS$.

**Area is multi-grid convergent.**
Having a **discrete representation** of a real object, digitized in an integer grid, **estimate** its perimeter (length of its border) with as small error as possible.

**One approach**

**Local polygonalization**
Approximate the object perimeter with the perimeter of a locally defined polygon.

Direct use of the perimeter of the polygon gives, on average, an **overestimate**.

Local step lengths can be chosen to reach better estimates.
How to assign local step lengths

Edge 1.08 times longer than true edge.

Using $a = 1, b = \sqrt{2}$ leads to an overestimate.
Error minimization

- Decide what error to minimize
  - The mean square error (MSE) minimization leads to estimators that, on average, perform well for lines of all directions.
  - The maximal error minimization leads to estimator with a better “controllable” error.

- Compute **optimal step lengths** to minimize the chosen error measure when estimating the length of straight segments of arbitrary direction.

  - To minimize MSE: \( a = 0.9481 \) and \( b = 1.3408 \).
    Root Mean Square (RMS) Error is 2.33%.
  - To minimize MaxErr: \( a = 0.9604 \) and \( b = 1.3583 \).
    Maximal Error is 3.95%.

- The error does not tend to zero with increasing resolution.
Crisp segmentation and feature estimation

Crisp discrete object representations, especially at low spatial resolutions, put strong limitations to precision of feature estimation:

Area: 28.274
Perim: 18.850

Area: 31.000
Perim: 19.422

Area: 26.000
Perim: 17.526

Area: 28.000
Perim: 18.867

Area: 27.000
Perim: 19.422

Area: 28.000
Perim: 18.311

Area: 28.000
Perim: 18.867

Area: 27.000
Perim: 18.867
The blame is not only on low resolutions...

Loss of information due to crisp decisions during object segmentation!

**Aim:** To utilize representations which enable information preservation.
Outline

1. Analysis of digital shapes: Crisp approach

2. Introducing uncertainty: Fuzzy approach

3. A special case: Coverage model

4. Concluding remarks
An example

Two different continuous disks with same crisp discrete representation

... and an alternative approach:
A fuzzy set of a reference set is a set of ordered pairs

\[ F = \{(x, \mu_F(x)) \mid x \in X\}, \]

where the membership function \( \mu_F : X \rightarrow [0, 1] \) indicates, for each element \( x \in X \), to what extent it belongs to the fuzzy set \( F \).

Observations:

- A crisp set is a special case of fuzzy set, where membership function takes only two values, 0 and 1.
- Membership function allows grading of an extent to which an element of a reference set belongs to the fuzzy set.
**The Fuzzy Approach**

- **Do not throw away information by making crisp decisions.**
- Allow *uncertainty* when performing segmentation; pixels do not belong exclusively to only one image component.
The Fuzzy Approach

• Do not throw away information by making crisp decisions.
• Allow uncertainty when performing segmentation; pixels do not belong exclusively to only one image component.

A standard image analysis task and its fuzzy solution

1. Sample preparation and Imaging
2. Pre-processing (optional)
3. Segmentation
   • Fuzzy segmentation - define suitable criteria and membership function
4. Feature extraction
   • Utilize existing theory and define feature estimators applicable to new representations
5. Classification, statistical evaluation, …
An \( \alpha \text{-cut} \) of a fuzzy set \( S \) is a \textbf{crisp set} \( ^\alpha S \) that contains all the elements in \( X \) that have membership value in \( S \) greater than or equal to \( \alpha \).

\[
^\alpha S = \{ x \mid S(x) \geq \alpha \}
\]

Given a function \( f : \mathcal{P}(X) \rightarrow \mathbb{R} \).

We can extend this function to \( f : \mathcal{F}(X) \rightarrow \mathbb{R} \), using the following equation

\[
f(S) = \int_0^1 f(\alpha S) \, d\alpha.
\]

This definition provides consistency with the crisp case.
The **area** of a fuzzy set $S$ on $X \subseteq \mathbb{R}$ is

$$\text{area}(S) = \int_{0}^{1} \text{area}(^\alpha S) \, d\alpha$$

$$= \int \int_{X} S(x, y) \, dxdy$$

or, for a **discrete and quantized** fuzzy set $S$ on $X \subseteq \mathbb{Z}^2$, with level set $\Lambda(A) = \{\alpha_1, \alpha_2...\alpha_n\}$

$$\text{area}(S) = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) \cdot \text{area}(^\alpha S) ,$$

$$= \sum_{X} S(x, y)$$

where $\alpha_{0} = 0$
The **perimeter** of a fuzzy set $S$

$$\text{perimeter}(S) = \int_0^1 \text{perimeter}(\alpha S) \, d\alpha$$

or, for a **discrete and quantized** fuzzy set $S$ on $X \subseteq \mathbb{Z}^2$, with level set $\Lambda(A) = \{\alpha_1, \alpha_2...\alpha_n\}$

$$\text{perimeter}(S) = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) \cdot \text{perimeter}(\alpha_i S).$$

where $\alpha_0 = 0$
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Pixel coverage digitization

The membership value of a pixel is equal to its relative coverage by an imaged object (here a part of a disk).

- Centre-point digitization assigns value 0 to the observed pixel.
- By coverage digitization assigned value is 0.3271.
- If (10-level) quantized coverage digitization is used, assigned value is 0.3.
Definition (non-quantized case)

For a given continuous object $S \subset \mathbb{R}^2$, inscribed into an integer grid with pixels $p(i,j)$, the *pixel coverage digitization* of $S$ is

$$
\mathcal{D}(S) = \left\{ \left( i, j, \frac{A(p(i,j) \cap S)}{A(p(i,j))} \right) \middle| (i, j) \in \mathbb{Z}^2 \right\},
$$

where $A(X)$ denotes the area of a set $X$.

- Keep good sides of fuzzy; stay close to the digital image, high information content, soft boundaries, robustness.
- Restrict to **one single meaning of memberships**; clear unique interpretation, enabling stronger theoretical results.
Feature extraction revisited

Precise shape analysis

Nataša Sladoje

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Concluding remarks

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<td><img src="image12.png" alt="Image 12" /></td>
<td>28.274</td>
<td>18.692</td>
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Area estimation

Definition

Area of a fuzzy segmented object $S$ defined by a membership function $S(i,j)$ on an integer grid is

$$A(S) = \sum_i \sum_j S(i,j),$$

where $(i,j)$ are points in the (integer) sampling grid.

Note

- Counting foreground pixels to estimate area in the crisp case is the same as summing membership values ($0$, or $1$) over the image.
- Summing membership values over the image leads to the same result as integrating area estimates over $\alpha$-cuts.
Area - error estimation

Crisp representation - reminder:

**Theorem**

*Area of a closed bounded set* $S$, *digitized in a grid with resolution* $r$ (*the number of grid points per unit*), *can be estimated by*

\[
\text{Area}(S) = \frac{1}{r^2} \tilde{\text{Area}}(rS) + \mathcal{O}\left(\frac{1}{r}\right).
\]

As already stated, $rS$ denote a scaling of the continuous set $S$ about the origin by the factor $r$. Scaling of the object can be used instead of changing resolution of a grid.

$	ilde{\text{Area}}(rS)$ is the **estimate of the area** of $rS$; it is equal to the number of pixels in discrete representation of $rS$. 
Coverage representation:

**Theorem**

*Area of a closed and bounded 2D shape $S$ can be estimated by*

$$\text{Area}(S) = \frac{1}{r_s^2} \tilde{A}(r_s S) + \mathcal{O}\left(\frac{1}{r_s^2}\right) + \mathcal{O}\left(\frac{1}{r_s r_f}\right)$$

*where $\tilde{A}$ is area of $r_s S$ computed from its coverage representation utilizing $r_f^2$ quantization (membership) levels.*
Area - error estimation

Coverage representation:

**Theorem**

*Area of a closed and bounded 2D shape $S$ can be estimated by*

\[
\text{Area}(S) = \frac{1}{r_s^2} \tilde{A}(r_s S) + O \left( \frac{1}{r_s^2} \right) + O \left( \frac{1}{r_s r_f} \right)
\]

*where $\tilde{A}$ is area of $r_s S$ computed from its coverage representation utilizing $r_f^2$ quantization (membership) levels.*

Utilizing $r_f^2$ membership levels in coverage representation provides (asymptotically) same precision in area estimation as increasing spatial resolution $r_f$ times.
Perimeter estimation
The straight edge of a halfplane

\[ \hat{l} = \gamma_5 * 4.49 \]

Example illustrating edge length estimation based on the difference \( d_c \) of column sums \( s_c \) for a segment \( (N = 4) \) of a halfplane edge given by \( y \leq 0.45x + 0.78 \).

\[
\begin{align*}
\hat{s}_c & : 1.20 \quad 1.80 \quad 2.20 \quad 2.60 \quad 3.20 \\
\hat{d}_c & : 0.60 \quad 0.40 \quad 0.40 \quad 0.60 \\
\hat{l}_c & : 1.17 \quad 1.08 \quad 1.08 \quad 1.17
\end{align*}
\]

\[
s_c = \sum_{j \geq 0} I(c, j), \quad d_c = s_{c+1} - s_c, \quad l_c = \sqrt{1 + d_c^2}
\]
# Length of the straight edge of a halfplane

## Discrete, grey-scale, quantized

Observe a halfplane $H = \{(x, y) \mid y(x) \leq kx + m, \ k, m \in [0, 1]\}$, over an interval $x \in [0, N]$, $N \in \mathbb{Z}^+$. Let $I$ be the quantized pixel coverage digitization of $H$.

Then

$$\hat{l} = \sum_{c=0}^{N-1} \gamma_n \sqrt{1 + d_c^2}.$$  

provides an estimate of the edge length $l$.

For a given number of (non-zero) grey-levels $n$, the scaling factor $\gamma_n < 1$ can be computed according to a derived formula.
Length of the straight edge of a halfplane

Discrete, grey-scale, quantized

Observe a halfplane \( H = \{ (x, y) \mid y(x) \leq kx + m, \ k, m \in [0, 1] \} \), over an interval \( x \in [0, N], \ N \in \mathbb{Z}^+ \).
Let \( I \) be the quantized pixel coverage digitization of \( H \).
Then
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\]
provides an estimate of the edge length \( l \).

For a given number of (non-zero) grey-levels \( n \), the scaling factor \( \gamma_n < 1 \) can be computed according to a derived formula.

Estimation error

The estimation error is a function of the number of grey-levels \( n \): 
\[
|\varepsilon_n| = \mathcal{O} \left( \frac{1}{n^2} \right).
\]
Relative errors in percent for test shapes digitized at increasing resolution for 5 different quantization levels and non-quantized \((n = \infty)\).
Coverage model in the presence of noise

- Noise can have strong influence on segmentation, and therefore can affect all subsequent analysis results.
- If we allow uncertainty, and utilize fuzzy approaches, we can decrease sensitivity to noise.
- The number of grey levels used to represent grading between “yes” and “no” can be tuned to provide good robustness of the methods.

- Area estimates based on coverage representation with up to $35\%$ of added Gaussian noise perform better than estimates based on noise-free crisp representation.
- Perimeter estimates based on coverage representation with up to $20\%$ of added Gaussian noise perform better than estimates based on noise-free crisp representation.
Summary and Conclusions

- During crisp segmentation, based on crisp (“yes”, or “no”) decisions, considerable information about continuous imaged objects is lost.
- This affects both accuracy and precision of estimated features.
- If uncertainty is allowed (“yes”, “no”, and “partly”), significant amount of information can be preserved in object representation.
- **Fuzzy set theory** provides a good foundation for handling this type of uncertainty.
- Additional information, stored as grey levels, can be exploited for more precise feature estimation.
- **Sensitivity to noise** can be reduced.
- With some assumptions on a membership function, theoretical error analysis of the estimators can be performed.
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- Sensitivity to noise can be reduced.
- With some assumptions on a membership function, theoretical error analysis of the estimators can be performed.

So, YES, uncertainty can help to measure more precisely!