Lab exercise: Ordinary differential equations, part 2

**Numeric stability and stiff differential equations**

In this part of the lab you will look at the concepts ‘numeric stability’ and how that concept is linked to so called stiff differential equations. You will also be able to see the difference between how explicit and implicit methods deal with these problems.

**Brief introduction**

If a numeric solution to a ODE is growing, while the exact solution decrease, the numerical solution is said to be unstable. It turns out that for certain types of differential equations, so called stiff differential equations, these kinds of problems arise for many numerical methods. Generally, implicit methods have significantly better stability properties than explicit methods, and this is something you will see in this part of the lab.

In the numerical experiments, we will use the differential equation

\[
\begin{aligned}
    \frac{dy}{dt} &= -\lambda y, \\
    y(0) &= y_0
\end{aligned}
\]

where \( \lambda \) is a positive constant. For larger \( \lambda \) the equation becomes increasingly stiff.

The analytical (exact) solution to the problem is \( y_0 e^{-\lambda t} \). As it is is known, it is possible to compare numerical and exact solution. We will here examine the numerical solutions we get from explicit and implicit Euler, respectively. These methods have the same order of accuracy, i.e. for a given step-size they give solutions with similar accuracy.

**To do**

1. **Explicit Euler method and Implicit Euler**
   Download **StabDemo.m** from the course page. Read the help text to understand how it is used. The program computes numerical solutions to the ODE problem described above, both using explicit and implicit Euler, and the solutions is compared with the exact solution.
   Run the program for \( \lambda = 50 \) and vary the stepsize. Explicit Euler has a certain criterion for stability, whereas implicit Euler is unconditionally stable. Choose stepsizes \( h \) both smaller and larger than the stability criterion. Also, choose stepsize equal to the criterion.
   Make the ODE more stiff by increasing \( \lambda \) to e.g. 100 and higher. Study what effect it has on the solution and on the restriction on \( h \).

**To think about**

- **How the exact solution changes when the problem gets more stiff.**
  What characterize a stiff problem?
- **What do we mean by numerically stable/unstable?**
- **Explicit methods, e.g. ode45, is a first-hand choice when solving ODEs.**
  Why don't we always use implicit methods (taking their better stability properties into account)?