Second Exam for the Course

Constraint Technology for Solving Combinatorial Problems

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Cover Sheet

This is a closed book exam, no written or printed material is allowed. This cover sheet should be handed in together with the exam. A mark from 50% to 84% earns a G passing grade, and a mark from 85% to 100% earns a VG passing grade. For each answer, you *must* show how you reached the result if nothing else is stated in the question. Unreadable or unclear answers will obtain 0 points. Each problem should be answered on a separate sheet. Write your name on each sheet. Circle below which questions you have answered.

Problem no.	Solution provided	Max	Your points
1	yes/no	8	
2	yes/no	12	
3	yes/no	22	
4	yes/no	18	
5	yes/no	20	
	Total:	80	

Name : ....

Pers.no. : ....

– Modelling Instructions –

Basically, your models should be clear and understandable in such a way that your classmates could understand and implement it with little difficulty.

You may write CSP models in the language of your choice such as Koalog, SICStus Prolog or FaCiLe. You may also write in pseudocode as you have seen during the lectures but please do not use any "magic" constraints that are not available in any of the solvers you have seen. You may use standard mathematical and logic notations, such as $M[i,j]$ (the element in row i and column j of the matrix M), $\sum$ (sum), $\forall i \in S$ (for all i in S), $\exists i \in S$ (there exists i in S), & (and), $\vee$ (or), $\rightarrow$ (implies), and so on. The instance data, as well as the decision variables and their domains must be declared, possibly in mathematical notation. Use only individual variables (whose domains are sets), do not use set variables (whose domains are sets of sets).

– Questions –

1. **Constraint Satisfaction Problems (8 Points)**

   Consider the CSP $P = \langle \mathcal{C}; \mathcal{DE} \rangle$ where

   - $\mathcal{C} = \{x_1 + x_2 > x_3 + x_4, x_1 \neq x_2, x_2 \neq x_3, x_3 \neq x_4\}$, and
   - $\mathcal{DE} = \langle x_1 \in \{1,2\}, x_2 \in \{2,3\}, x_3 \in \{2,3\}, x_4 \in \{1,2\} \rangle$

   (a) What is the solution set of $x_1 \neq x_2$? (**2 Points**)

   (b) What is the solution set of $x_1 + x_2 > x_3 + x_4$? (**2 Points**)

   (c) What is the solution set of P? (**2 Points**)

   (d) Give a constraint c such that the solution set of $P' = \langle \mathcal{C} \cup c \; ; \; \mathcal{DE} \rangle$:

   i. contains only one element. (**1 Point**)

   ii. is empty. (**1 Point**)

2. **Consistency (12 Points)**

   (a) Define *directional arc consistency* for CSPs with binary constraints. (**4 Points**)

   (b) Construct a binary CSP P such that

   - after applying *directional arc consistency* nothing is done to the domains of the variables in P.
   - after applying *arc consistency* the domain of at least one of the variables in P is changed. (**2 Points**)

   (c) Consider the CSP $P = \langle \mathcal{C}; \mathcal{DE} \rangle$ where:

   - $\mathcal{C} = \{x + y < z, x < y, \textit{all\_different}(\{x,y,z\})\}$, and
   - $\mathcal{DE} = \langle x \in \{1,\ldots,5\}, y \in \{1,\ldots,5\}, z \in \{1,\ldots,7\} \rangle$

   i. After applying *hyper arc consistency*, what will the domains of the variables be? (**3 Points**)

   ii. What is the width of P? (**1 Point**)

   (d) Is the following CSP

   $$\langle x - y \leq 4, y - z \leq 5, x - z \leq 10 \; ; \; x \in \{0,\ldots,10\}, y \in \{0,\ldots,10\}, z \in \{0,\ldots,10\} \rangle$$

   path consistent? Why or why not? (**2 Points**)

3. **Modelling** (**22 Points**)

(a) In this question you will model a Swedish instance of the *travelling salesman problem (TSP)*. Given is a set of Swedish cities and a table with distances between any two cities. The problem is to find a cyclic path starting from and ending in Stockholm, visiting all other cities exactly once, such that the total distance is minimised.

The cities in our tour are Mora, Göteborg, Stockholm, Sundsvall, Umeå, Västerås and Östersund. The distances between the cities are given below:

	Mora	Göteborg	Stockholm	Sundsvall	Umeå	Västerås	Östersund
Mora	0	471	315	277	547	228	305
Göteborg	471	0	478	720	989	378	775
Stockholm	315	478	0	394	663	113	552
Sundsvall	277	720	394	0	270	367	186
Umeå	547	989	663	270	0	636	365
Västerås	228	378	113	367	636	0	505
Östersund	305	775	552	186	365	505	0

Your task is to come up with a constraint based model that would find a minimal tour of this particular TSP instance once implemented in a constraint solver. The following combinatorial constraints may turn out to be useful:

- *element*$(x, \langle z_1, \ldots, z_n \rangle, y)$, where x, $z_1, \ldots, z_n$ and y are variables or integers, enforces that y is equal to the xth element of the sequence $\langle x_1, \ldots, x_n \rangle$.
- *circuit*$(\langle x_1, \ldots, x_n \rangle)$, where $x_1, \ldots, x_n$ are variables or integers with the domain $\{1, \ldots, n\}$, enforces that the nodes $1, \ldots, n$ in a graph form a a cyclic path such that x_i is the successor of i in the path. For example, the instance of the constraint *circuit*$(\langle 3, 4, 2, 1 \rangle)$ corresponds to the cycle $1 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 1$. (**7 Points**)

(b) Consider the following problem statement:

> The coordinator of a golf club has the following problem. In her club, there are 32 golfers, each of whom play golf once a week, and always in 8 groups of size 4. She would like to come up with a schedule of play for these golfers, to last for 10 weeks, such that no golfer plays in the same group as any other golfer on more than one occasion.

We may generalise this to the following:

> n golfers want to play golf once a week organized in g groups of size s (hence $n = g * s$). Find a schedule of play for the golfers lasting for w weeks such that each golfer plays once a week and such that no pair of golfers play in the same group more than once.

i. Come up with a constraint based model for this problem that, given the constants g, s and w, finds a schedule of play (if it exists) for the n golfers respecting the constraints above. (**7 Points**)

ii. Identify the (possible) symmetries in your model. (**4 Points**)

iii. Try to break as many of these symmetries as possible. (You may assume a global constraint *lex*$(\langle x_1, \ldots, x_n \rangle, \langle y_1, \ldots, y_n \rangle)$ that enforces that the sequence $\langle x_1, \ldots, x_n \rangle$ is lexicographically smaller than the sequence $\langle y_1, \ldots, y_n \rangle$. (**4 Points**)

4. **Search (18 Points)**

(a) Consider the CSP $P = \langle x + y < 2, z < y \ ; \ x \in \{0,1,2\}, y \in \{0,1,2\}, z \in \{0,1,2\}\rangle$ and the ordering on the variables $x \prec y \prec z$.

i. Draw the *partial look ahead* search tree of P. (**3 Points**)

ii. Draw the *full look ahead* search tree of P. (**3 Points**)

(b) Consider the COP defined by the CSP $P = \langle \mathcal{C}; \mathcal{DE}\rangle$ and the objective function $obj : Sol(P) \to \mathbb{R}$ where:

- $\mathcal{C} = \{x_1 \neq x_2, x_2 = x_3, x_3 > x_4, x_4 \neq x_5\}$, and
- $\mathcal{DE} = \langle x_1 \in \{4,5\}, x_2 \in \{3,5\}, x_3 \in \{3,5\}, x_4 \in \{2,3\}, x_5 \in \{1,3\}\rangle$, and
- $obj(d_1, \ldots, d_5) = d_1 - d_2 + d_3 - d_4 + d_5$.

i. A *heuristic function* in branch and bound search must satisfy certain properties. State these properties and discuss briefly why they are important. (Assume the *maximisation* version of branch and bound.) (**4 Points**)

ii. Assume that we want to find a solution to P that *maximises* the function *obj* by using branch and bound search. Define a heuristic function $h : \wp(D_1) \times \cdots \times \wp(D_5) \to \mathbb{R}$ for the search, where D_i is the domain of the variable x_i. (**2 Points**)

iii. Use the heuristic function defined above to draw the branch and bound search tree for the COP. Your tree should be drawn with respect to the following instructions:

- Enumerate the variables in the order $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$.
- Assign values to the variables starting with their *highest* value first.
- There should be no propagation, i.e., a constraint such as $x < y$ may only fail when both x and y are assigned. (And of course only if $x \geq y$ under that assignment.)
- Indicate in the tree
  - where a branch is cut because of a failed constraint.
  - where a branch is cut because the branch and bound search cannot find an optimal solution in it.
  - where the optimal solution is found. (**6 Points**)

5. **AllDifferent Filtering Algorithm (20 Points)**

Consider the variables with their respective domains $x_0 \in \{0, 1, 2\}$, $x_1 \in \{0, 1, 2\}$, $x_2 \in \{0, 1, 2\}$, $x_3 \in \{2, 3\}$, $x_4 \in \{0, \ldots, 5\}$, and $x_5 \in \{5, 6\}$. Your task in this question is to go through one filtering of Régin's algorithm for the constraint *all_different*$(\{x_0, x_1, x_2, x_3, x_4, x_5\})$.

(a) Draw the variable value graph for the constraint *all_different*$(\{x_0, x_1, x_2, x_3, x_4, x_5\})$. (**3 Points**)

(b) Draw the graph in (a) again but this time orient the edges (give them arrows) with respect to a maximal matching of your choice. (**3 Points**)

(c) Draw the graph in (b) again but this time show only the edges that take part in an alternating path (mark these with a 'p') or an alternating cycle (mark these with a 'c'). (**5 Points**)

(d) Draw the graph in (a) again but this time remove the edges that cannot take part in a solution with respect to Régin's algorithm. (**3 Points**)

(e) What are the domains of the variables after the filtering you just went through? (**2 Points**)

(f) What main result from matching theory is Régin's algorithm based on? In what way is this used in the algorithm? (**4 Points**)

Good Luck!