First Exam for Course 1DL118:

Constraint Technology for Solving Combinatorial Problems

Spring 2006

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Cover Sheet

Materials: This is an individual, closed-world exam: no written or printed material (except dictionaries) is allowed, nor any electronic devices. Preferably use a pencil.

Grading: A mark from 80% to 100% of the maximum earns a high passing grade (a Swedish VG for local students, respectively an ECTS A or B for visiting students). A mark from 50% to 79% of the maximum earns a lower passing grade (a Swedish G for local students, respectively an ECTS C, D, or E for visiting students). A mark from 0% to 49% of the maximum earns a failure grade (a Swedish U for local students, respectively an ECTS FX or F for visiting students). The distinction between the ECTS grades will be made according to the recommendations of Uppsala University.

Help: Normally, an instructor will attend this exam from 11:00 to 12:00.

Answers: Your answers should be written in English. In each answer, you *must* show how you reached your result, unless stated otherwise in the question. Unreadable or unclear answers will get zero points. The answer to each question should be started on a new sheet; you can use both sides of each sheet. Write your name on each answer sheet. This cover sheet should be handed in together with your solutions. Circle below which questions you have actually addressed.

Problem	Solution Provided?	Points	Your Points
1	yes/no	8	
2	yes/no	11	
3	yes/no	15	
4	yes/no	16	
5	yes/no	22	
	Total:	72	

Identity: Name & Civic Registration Number: ....

1. **Constraint Satisfaction Problems**

   Consider the CSP $P = \langle V, U, C \rangle$ where

   - $V = \{x_1, x_2, x_3, x_4\}$
   - $U = \{1, 2, 3\}$
   - $C = \{x_1 + x_2 > 2 \cdot x_3 + x_4, x_1 \neq x_2, x_2 \neq x_3, x_3 \neq x_4\}$

   (a) What is the set of solutions of $x_1 \neq x_2$? (**2 points**)

   (b) What is the set of solutions of $x_1 + x_2 > 2 \cdot x_3 + x_4$? (**2 points**)

   (c) What is the set of solutions of P? (**2 points**)

   (d) Give a constraint c such that the set of solutions of $P' = \langle V, U, C \cup \{c\} \rangle$:

   i. contains only one element. (**1 point**)

   ii. is empty. (**1 point**)

2. **Consistency**

   (a) Consider the following CSP:

   $$\langle x < y, x + y > 4, y \leq z; x \in \{2, \ldots, 10\}, y \in \{0, \ldots, 10\}, z \in \{0, \ldots, 10\} \rangle$$

   After applying arc consistency, what will the domains of the variables be? (**2 points**)

   (b) Define what it means for a CSP to be hyper-arc-consistent? (**2 points**)

   (c) Consider the following CSP:

   $$\langle x + y < z, x < y, \mathit{distinct}(\{x, y, z\}); x \in \{1, \ldots, 5\}, y \in \{1, \ldots, 5\}, z \in \{1, \ldots, 7\} \rangle$$

   After applying hyper arc consistency, what will the domains of the variables be? (**3 points**)

   (d) Consider the following CSP:

   $$\langle x = 3y + 5z; x \in \{0, 1, 2, 3, 4, 5, 6, 7\}, y \in \{0, 1, 2\}, z \in \{-1, 0, 1, 2\} \rangle$$

   After applying hyper arc consistency, what will the domains of the variables be? (**4 points**)

3. **Search and Propagation**

   (a) Prove that the following is a propagator (recall that you must prove that it is *contracting* and *monotonic*):

   $$p(s) = \left\{ \begin{array}{l} x \mapsto s(x) \cap s(y), \\ y \mapsto s(y) \cap \{1, 2\} \end{array} \right\}$$

   (**4 points**)

   (b) Consider an arbitrary universe U. Is the following function a propagator?

   $$p(s) = \{x \mapsto s(x) \cap (U - s(y)), y \mapsto s(y) \cap (U - s(x))\}$$

   If it is not a propagator, then you must show why, otherwise you must provide a proof. (**4 points**)

   (c) Define what it means for a propagator to implement a constraint. You must state all the conditions that it must satisfy and give an example. (**2 points**)

   (d) How are events used in a constraint solver to do propagation more efficiently? (**2 points**)

   (e) A branching is a function $b(P, s)$ that takes a set of propagators P and a store s and returns a tuple of finite propagator sets $(P_1, \ldots, P_n)$. State what properties a branching must satisfy. (**3 points**)

4. **Filtering Algorithm for the *distinct* Global Constraint**

   Consider the variables with their respective domains $x_1 \in \{1, 2, 3\}$, $x_2 \in \{2, 3\}$, $x_3 \in \{2, 3\}$, $x_4 \in \{3, 4, 5\}$, and $x_5 \in \{5, 6\}$. Go through one filtering by Régin's algorithm for the constraint $\mathit{distinct}([x_1, x_2, x_3, x_4, x_5])$:

   (a) Draw the variable value graph for the constraint $\mathit{distinct}([x_1, x_2, x_3, x_4, x_5])$. (**3 points**)

   (b) Draw the graph in (a) again, but this time orient the edges (give them arrows) with respect to a maximal matching of your choice. (**2 points**)

   (c) Draw the graph in (b) again, but this time show only the edges that take part in an alternating path (mark these with a 'p') or an alternating cycle (mark these with a 'c'). (**4 points**)

   (d) Draw the graph in (a) again, but this time remove the edges that cannot take part in a solution according to Régin's algorithm. (**2 points**)

   (e) What are the domains of the variables after this filtering? (**1 point**)

   (f) What main result from matching theory is Régin's algorithm based on? In what way is this used in the algorithm? (**4 points**)

5. **Modelling:** Your models should be clear and comprehensible, say such that your classmates can understand and implement them without difficulty. You may write models in the language of your choice among Gecode/J, SICStus Prolog, and FaCiLe. It is actually recommended that you write them in pseudocode, as on the lecture slides. *The instance data, as well as the decision variables and their domains must be declared*, possibly in mathematical notation, *and their meanings must be given.* You may use standard mathematical and logical notations, such as $M[i,j]$ (the element in row i and column j of the matrix M; *you can only use this notation when each index is a constant*, or $\star$ in case you want to extract a slice of the matrix), $\sum_{i \in S} e(i)$ (the sum over all i in the set/enumeration/range S of the numerical expressions $e(i)$), $\forall i \in S \,.\, \alpha(i)$ (for all i in S, the formula $\alpha(i)$ is true), $\wedge$ (and), $\leftrightarrow$ (is equivalent to; use this for reification or channelling), $\rightarrow$ (implies; use this for channelling), and so on. Try hard to avoid $\vee$ (or). You may not use $\exists i \in S \,.\, \alpha(i)$ (there exists an i in S such that $\alpha(i)$ is true), nor $\exists! i \in S \,.\, \alpha(i)$ (there exists exactly one i in S such that $\alpha(i)$ is true), nor negation. You need not give any search statements. Only use the following global constraints:

- $\mathit{circuit}(\langle x_1, \ldots, x_n \rangle)$, where $x_1, \ldots, x_n$ are integers or variables with the domain $\{1, \ldots, n\}$, enforces that the nodes called $1, \ldots, n$ in a graph form a cyclic path such that node x_i is the successor of node i in the path. For example, the instance $\mathit{circuit}(\langle 3, 4, 2, 1 \rangle)$ corresponds to the cycle $1 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 1$.
- $\mathit{element}(\langle i_1, \ldots, i_n \rangle, x, y)$, where $i_1, \ldots, i_n, x, y$ are integers or variables, enforces that y is equal to the x^{th} element of the sequence $\langle i_1, \ldots, i_n \rangle$.
- $\mathit{gcc}(\langle x_1, \ldots, x_n \rangle, \langle v_1, \ldots, v_m \rangle, \langle \mathit{min}_1, \ldots, \mathit{min}_m \rangle, \langle \mathit{max}_1, \ldots, \mathit{max}_m \rangle)$ enforces that the number of variables in the sequence $\langle x_1, \ldots, x_n \rangle$ that take the constant value v_j is between the integer constants min_j and max_j inclusive, for all $j \in 1, \ldots, m$.
- $\mathit{lex}(\langle x_1, \ldots, x_n \rangle, \langle y_1, \ldots, y_n \rangle)$ enforces that the variable sequence $\langle x_1, \ldots, x_n \rangle$ is lexicographically smaller than or equal to the variable sequence $\langle y_1, \ldots, y_n \rangle$.

Answer the following sub-questions:

(a) Consider a Swedish instance of the *travelling salesperson problem* (TSP). Given are seven Swedish cities, namely Mora, Göteborg, Stockholm, Sundsvall, Umeå, Västerås, and Östersund, and the following table D with their travel distances:

	Mora	Göteborg	Stockholm	Sundsvall	Umeå	Västerås	Östersund
Mora	0	471	315	277	547	228	305
Göteborg	471	0	478	720	989	378	775
Stockholm	315	478	0	394	663	113	552
Sundsvall	277	720	394	0	270	367	186
Umeå	547	989	663	270	0	636	365
Västerås	228	378	113	367	636	0	505
Östersund	305	775	552	186	365	505	0

Model the optimisation problem of finding a path across Sweden such that:

(cyclic-path) The path starts from and ends in the same city.

(visit-once) Every city is visited (reached) exactly once.

(objective) The total travel distance is minimised.

Relate your constraints to the constraint names given above. (**6 points**)

(b) Consider the *social golfer problem*: The members of a golf club want to play golf once a week for w weeks, always in g groups of size s (hence there are $g \cdot s$ golfers). Find a schedule of play such that the following constraints are satisfied:

(constant-group-size) Every week, each group is of the given constant size s.

(play-once-a-week) Each golfer plays exactly once a week (in some group).

(meet-at-most-once) Any two distinct golfers play together in at most one week.

Answer the following:

i. Model this problem. *Use only scalar variables* (whose domains are sets or ranges), *but not set variables* (whose domains are sets of sets). Show how some, if any, of the constraints named above are automatically enforced by your choice of decision variables. (Hint: Try to achieve this for at least one named constraint, via a two-dimensional matrix of scalar decision variables.) Relate each of your constraints to one of the constraint names given above, or declare it to be a channelling constraint. (Hint: Recall that with reification one can transform disjunction into integer inequality.) (**11 points**)

ii. Identify the variable and value symmetries in your model. Show how some, if any, of the symmetries of the problem are broken by your model. (**3 points**)

iii. Break as many of the symmetries of your model as reasonable. (**2 points**)

Good Luck!