10. Loop shaping

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LOOP SHAPING

Open loop system

\[ y = Gu \]

Feedback

\[ u = -F_y y \]

Make the loop gain \( GF_y \), the sensitivity function \( S \), the complementary sensitivity function \( T \), and the disturbance to input transfer function \( G_{wu} \) appropriate.

\[
S = (I + GF_y)^{-1}
\]

\[
T = I - S = (I + GF_y)^{-1}GF_y
\]

\[
G_{wu} = -(I + F_y G)^{-1}F_y
\]

\[
- F_y (I + GF_y)^{-1} - F_y S
\]

LOOP SHAPING, cont’d

The transfer functions \( S, T, G_{wu} \) can not all be small for all frequencies.

Introduce weighting \( W_S, W_T, W_u \) and require that

\[
W_S(i\omega)S(i\omega), \ W_T(i\omega)T(i\omega), \ W_u(i\omega)G_{wu}(i\omega)
\]

are small for all frequencies.

Assume \( G \) and \( W_S \) to be strictly proper (without direct term).

Extended system description

Inputs \( u, w \)

Outputs \( z_1 = W_u u, \ z_2 = W_T Gu, \ z_3 = W_S (Gu + w), \ y = Gu + w \)

\[ W_u \rightarrow z_1 \]

\[ u \]

\[ G \]

\[ W_T \rightarrow z_2 \]

\[ W_S \rightarrow z_3 \]

\[ W_S \]

\[ G \]

\[ W_T \]

\[ z_2 \]

\[ z_3 \]

\[ y \]
LOOP SHAPING, cont’d

With feedback $\nu = -F_y y$, the closed loop system becomes

$$
\begin{pmatrix}
  z_1 \\
  z_2 \\
  z_3 
\end{pmatrix} = \begin{pmatrix}
  W_u G_{wu} \\
  -W_T T \\
  W_S S 
\end{pmatrix} \begin{pmatrix}
  \Delta \\
  w \equiv G_{ec} w
\end{pmatrix}
$$

LOOP SHAPING, cont’d

State space model for the open-loop system $(u, w) \mapsto (z, y)$.
Assume innovations form (crucial!) is given

$$
\begin{align*}
\dot{x} &= A\dot{x} + Bu + Nw \\
z &= M\dot{x} + Du \\
y &= C\dot{x} + w
\end{align*}
$$

Comments:

1. Innovations form implies $A - NC$ is stable.
   [Possibly make a transformation of $u$ to achieve this; it is always possible if $D^T D$ is invertible, that is, if $\lim_{s \to \infty} W_u(s)$ is invertible.]

DESIGN APPROACHES

- Optimal $H_2$ control
- (Optimal) $H_\infty$ control
- Robust control
- LQG control

OPTIMAL $H_2$ CONTROL

Minimize the design criterion

$$
\begin{align*}
V(F_y) &= \frac{1}{2\pi} \int \left[ |W_S (i\omega) S(i\omega)|^2 + |W_T (i\omega) T(i\omega)|^2 + |W_u(i\omega) G_{wu}(i\omega)|^2 \right] d\omega \\
&= \frac{1}{2\pi} \int \left[ |G_{ec}(i\omega)|^2 \right] d\omega = \| G_{ec} \|^2
\end{align*}
$$
OPTIMAL $H_2$ CONTROL, cont’d

It holds that $z = G_{ee}w$. Let $w$ have intensity $I$. Then

$$\| z \|^2 = \frac{1}{2\pi} \text{tr} \left( \int \Phi_2(\omega) d\omega \right) = \frac{1}{2\pi} \int \text{tr}(\Phi(\omega)) d\omega = \frac{1}{2\pi} \int |G_{ee}(i\omega)|^2 d\omega = V(F_y)$$

Hence

$$V(F_y) = \| z \|^2 = \| Mx + Du \|^2 = \| Mx \|^2 + \| u \|^2$$

The optimal solution is given by the LQG theory!

OPTIMAL $H_2$ CONTROL, cont’d

Solution in state space form (recall open loop system given in innovations form)

$$\dot{x} = Ax + Bu + N(y - C\hat{x})$$
$$u = -L\hat{x}$$

with

$$L = B^T S$$
$$0 = A^T S + SA + M^T M - SBB^T S$$

Hence

$$F_y(s) = L(sI - A + BB^T S + NC)^{-1} N$$

EXAMPLE, LOOP SHAPING

A DC servo

$$G(s) = \frac{1}{s(s + 1)}$$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Weights

$$W_S(s) = 1/s, \quad W_u = 1, \quad W_T = \frac{5(s + 1)}{s + 10}$$

PERFORMANCE BOUNDS

$$\| W_S S \|_\infty \leq \gamma \Rightarrow |S(i\omega)| \leq \gamma |W_S(i\omega)|^{-1} = \gamma \omega$$

$$\| W_u G_{wu} \|_\infty \leq \gamma \Rightarrow |G_{wu}(i\omega)| \leq \gamma |W_u(i\omega)|^{-1} = \gamma$$

$$\| W_T T \|_\infty \leq \gamma \Rightarrow |T(i\omega)| \leq \gamma |W_T(i\omega)|^{-1} = \frac{\sqrt{\omega^2 + 100}}{5(\omega^2 + 1)}$$
**EXAMPLE, cont’d**

**Upper bounds**

![Graphs showing upper bounds](image)

**Extended model**

Basic model

\[ x = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \]

\[ y_o = \begin{pmatrix} 1 & 0 \end{pmatrix} x \]

Introduce further states

\[
\begin{align*}
z_1 & = u \\
z_2 & = W_T y_o \\
& = \frac{5(s + 1)}{s + 10} y_o = \left( 5 + \frac{-45}{s + 10} \right) y_o \\
z_3 & = W_S (y_o + w) \\
& = \frac{1}{s} (y_o + w)
\end{align*}
\]

Choose \( x_3 \) and \( x_4 \) as

\[ x_3 = z_3 \Rightarrow \dot{x}_3 = y_o + w \]

\[ x_4 = \frac{1}{s + 10} y_o \Rightarrow \dot{x}_4 = -10 x_4 + y_o \]

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**EXAMPLE, cont’d**

**Augmented model**

\[ x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -10 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u \]

\[ + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} w \]

\[ y = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x + w \]

\[ z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & -45 \\ 0 & 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u \]

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**EXAMPLE, cont’d**

**Input and output for a reference step**

![Step response graphs](image)
EXAMPLE, cont’d

Sensitivity functions and weights
\( (\gamma = 1 \text{ in plots}) \)

\[
\begin{align*}
\| G_{ec} \|_\infty &= \max_\omega \sigma(G_{ec}(i\omega)) \\
\| G_{ec} \|_\infty &\leq \gamma \\
|W_S(i\omega)S(i\omega)| &\leq \gamma, \quad \forall \omega \\
|W_T(i\omega)T(i\omega)| &\leq \gamma, \quad \forall \omega \\
|W_u(i\omega)G_{wu}(i\omega)| &\leq \gamma, \quad \forall \omega
\end{align*}
\]

OPTIMAL \( \mathcal{H}_\infty \) CONTROL

Design objective: Minimize
\[
\| G_{ec} \|_\infty = \max_\omega \sigma(G_{ec}(i\omega))
\]

Find regulators that satisfy
\[ \| G_{ec} \|_\infty \leq \gamma \]

Consequences
\[
\begin{align*}
|W_S(i\omega)S(i\omega)| &\leq \gamma, \quad \forall \omega \\
|W_T(i\omega)T(i\omega)| &\leq \gamma, \quad \forall \omega \\
|W_u(i\omega)G_{wu}(i\omega)| &\leq \gamma, \quad \forall \omega
\end{align*}
\]

OPTIMAL \( \mathcal{H}_\infty \) CONTROL, cont’d

Solution.
Assume
\[
A^T S + SA + M^T M + S(\gamma^{-2} N N^T - BB^T) S = 0
\]
has a positive semidefinite solution \( S = S_\gamma \), and that \( A - BB^T S_\gamma \) is stable.

Consider the regulator
\[
\begin{align*}
\dot{x} &= A\dot{x} + Bu + N(y - C\dot{x}) \\
u &= -L_{\infty} x
\end{align*}
\]
with \( L_{\infty} = B^T S_\gamma \). Then
\[
F_y(s) = L_{\infty} (sI - A + BB^T S_\gamma + NC)^{-1} N
\]

OPTIMAL \( \mathcal{H}_\infty \) CONTROL, cont’d

Results.

1. The bounds on
\[
|W_S(i\omega)S(i\omega)|, \quad |W_T(i\omega)T(i\omega)|, \quad |W_u(i\omega)G_{wu}(i\omega)|
\]
are satisfied.

2. If the Riccati equation has no positive semidefinite solution with \( A - BB^T S_\gamma \) stable, then there is no linear regulator satisfying the bound
\[
\| G_{ec} \|_\infty < \gamma
\]
**OPTIMAL $H_\infty$ CONTROL, cont’d**

**Design steps**

1. $G$ is given.
3. Choose a constant $\gamma$.
4. Solve the Riccati equation, compute $L_{\infty}$.
   (a) If no solution exists, increase $\gamma$ and go to Step 3.
   (b) If a solution exists, accept it, or decrease $\gamma$ and go to Step 3.
5. Check the properties of the closed loop system. If not acceptable, go to Step 2.

As a first attempt the weights $W_u$, $W_S$, $W_T$ are chosen of low order (say 1 or 2), and diagonal in the multivariable case.

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**Example - DC servo**

Weights as before

$$W_S = 1/s, \quad W_u = 1, \quad W_T = \frac{5(s + 1)}{s + 10}$$

Try different (decreasing) values of $\gamma$ (until the Riccati equation has no positive definite solution.)

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**Example, cont’d**

Response to step in the reference signal

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**Sensitivity functions and weights**
ROBUST LOOP SHAPING

Try to obtain good stability margins without degrading performance.
(Useful design approach for MIMO systems)

System \( (D = 0) \)
\[
\dot{x} = Ax + Bu \\
y = Cx
\]

Feedback
\[
u = -F_y \dot{y}
\]

Controller
\[
\dot{x} = Ax + Bu + K(y - C\dot{x}) \\
u = -L\dot{x}
\]

ROBUST LOOP SHAPING, cont’d

Design procedure (Glover-McFarlane)
1. Solve the Riccati equations
\[
AZ + ZA^T - ZC^T C Z + BB^T = 0 \\
A^T X + XA - XBB^T X + C^T C = 0
\]
2. Set
\[
\lambda_m = \max \ \text{eig}[XZ]
\]
3. Set (with \( \alpha \) a scaling factor, strictly larger than 1)
\[
\gamma = \alpha \sqrt{1 + \lambda_m} \\
R = I - \frac{1}{\gamma^2} (I + XZ) \\
L = B^T X \\
K = R^{-1} ZC^T
\]

ROBUST LOOP SHAPING, cont’d

Example – DC servo
\[
G(s) = \frac{1}{s(s + 1)}, \quad \alpha = 1.1
\]

Step response

- \[\gamma = \frac{1}{s(s + 1)}\]
EXAMPLE, ROBUST LOOP SHAPING, cont’d

Sensitivity functions

ROBUST LOOP SHAPING, cont’d

Design in the multivariable case

1. Use RGA to make $G(s)$ ‘as diagonal as possible’.
2. Choose $W_p(s)$ diagonal, so that $G(s)W_p(s)$
is approximately equal to the desired loop gain.
3. Possibly adjust constant matrices $W_1$ and
$W_2$ to achieve performance (crossover
frequency, decoupling, etc) of
$W_2G(s)W_p(s)W_1$.
4. Compute the robust controller $F_p(s)$. (Go
to Step 2 in case $\gamma$ is ‘large’, say if $\gamma \geq 4$).
5. Apply the feedback

$$u = -W_p(s)W_1F_p(s)W_2y$$

COMPARING DIFFERENT DESIGN TECHNIQUES

Compare the following techniques, applied to
the DC servo

- LQG
- LTR
- $H_2$
- $H_\infty$
- Robust control

COMPARING DIFFERENT DESIGN TECHNIQUES, cont’d

Step response