3. Linear systems – properties

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LINEAR MULTIVARIABLE SYSTEM

- Solution of the system equations
- Controllability and observability
- Poles and zeros
- Stability
- Frequency functions

Which concepts are the same as in the SISO case?

SOLUTION OF THE SYSTEM EQUATIONS

Continuous-time. System representation

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) + Du(t) \]

Solution

\[ x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \]

- First term gives influence of the initial condition
- Second term gives influence of the input actions over the time interval \([t_0, t]_\).
CONTROLLABILITY AND OBSERVABILITY

1. The state $x^*$ is controllable if there exists an input $u(t)$ that moves the system from $x = 0$ to $x = x^*$ in finite time.

2. A system is controllable if all states are controllable.

3. The state $x^*$ is non-observable if $x(0) = x^*$ and $u(t) \equiv 0$ gives $y(t) \equiv 0$.

4. A system is observable if there are no non-observable states.

N.B. The above definitions apply also in discrete-time.

CONTROLLABILITY AND OBSERVABILITY, cont’d

• The controllable states span the range of the controllability matrix

$$\mathcal{S}(A, B) = [B \ AB \ A^2B \ \ldots \ A^{n-1}B]$$

• The system is controllable $\Leftrightarrow \mathcal{S}(A, B)$ has full rank. (If there is one input only, $\mathcal{S}$ is a square matrix, and full rank is equivalent to non-singular.)

• A system in controller canonical form is always controllable.

CONTROLLABILITY AND OBSERVABILITY, cont’d

• The non-observable states span the null space of the observability matrix

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

• The system is observable $\Leftrightarrow \mathcal{O}(C, A)$ has full rank.

• A system in observer canonical form is always observable.

• (Pole placement) For given $A$ and $B$ one can find a matrix $L$ so that $A - BL$ has a given set of eigenvalues $\Leftrightarrow$ the matrix $\mathcal{S}(A, B)$ has full rank $\Leftrightarrow$ the system $\dot{x} = Ax + Bu$ is controllable.

• The system is a minimal realization if it is both controllable and observable.

• The system is controllable $\Leftrightarrow$

$$\begin{bmatrix} A - \lambda I & B \end{bmatrix} \text{ full rank } \forall \lambda$$

• The system is observable $\Leftrightarrow$

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \text{ full rank } \forall \lambda$$
CONTROLLABILITY AND OBSERVABILITY, cont’d

Adjusted concepts that concern unstable modes,

- The system \((A, B, C)\) is stabilizable if there exists a matrix \(L\) such that \(A - BL\) has all eigenvalues in the stability region.
- A system \((A, B, C)\) is stabilizable \(\iff\) all unstable modes are controllable.
- A system \((A, B, C)\) is detectable if there exists a matrix \(K\) such that \(A - KC\) has all eigenvalues in the stability region.
- A system \((A, B, C)\) is detectable \(\iff\) all non-observable modes are stable.

POLES

- The poles of a system are the eigenvalues of the system matrix \(A\).
- The system order = the number of state variables = \# of poles
- Nontrivial how to find the poles from \(G(s)\) in the multivariable case (the multiplicity is tricky):
  \textit{Definition}: The pole polynomial is the least common denominator to all minors [sub-determinants] of \(G(s)\).

POLES, EXAMPLE

\[
G(s) = \begin{pmatrix}
\frac{2}{s+1} & \frac{3}{s+2} \\
\frac{1}{s+1} & \frac{1}{s+1}
\end{pmatrix}
\]

Minors:

\[
\frac{2}{s+1}, \frac{3}{s+2}, \frac{1}{s+1}, \frac{2(s+1)}{(s+1)^2(s+2)}
\]

Least common denominator (pole polynomial)

\[
(s+1)^2(s+2)
\]

Poles

- \(s = -1\) (double pole)
- \(s = -2\)

POLES, EXAMPLE, cont’d

State space realization (Gilbert’s algorithm) [extension of diagonal form!]

\[
G(s) = \frac{1}{s+1} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} + \frac{1}{s+2} \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}
\]

\[
= \frac{1}{s+1} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} + \frac{1}{s+2} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}
\]

\[
\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} x + \begin{pmatrix} 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} u
\]

\[
y = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix} x
\]
ZEROS

How to define in the multivariable case?

- If \( m = p \), possibly the poles of \( G^{-1}(s) \).
- \textit{Definition:} The zero polynomial is the greatest common divisor for the numerators of the maximal minors [sub-determinants] of \( G(s) \), normalized so that they have the pole polynomial as denominator,.
- \textit{Result:} The zeros are the values of \( s \) that makes

\[
\begin{pmatrix}
sI - A & B \\
-C & D \\
\end{pmatrix}
\]

rank-deficient,

ZEROS, EXAMPLE, cont’d

\[
G(s) = \begin{pmatrix}
\frac{2}{s+1} & \frac{3}{s+2} \\
\frac{1}{s+1} & \frac{1}{s+2} \\
\end{pmatrix}
\]

Maximal minor:

\[
\det G(s) = \frac{-s + 1}{(s+1)^2(s+2)}
\]

Already normalized with the pole polynomial

Zero: \(-s + 1 = 0 \implies s = 1\).

ZEROS, EXAMPLE, cont’d

\[
G^{-1}(s) = \begin{pmatrix}
\frac{2}{s+1} & \frac{3}{s+2} \\
\frac{1}{s+1} & \frac{1}{s+2} \\
\end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix}
\frac{(s+1)(s+2)}{-s+1} & -3 \frac{(s+1)^2}{-s+1} \\
\frac{(s+1)(s+2)}{-s+1} & 2 \frac{(s+1)(s+2)}{-s+1} \\
\end{pmatrix}
\]

\( G^{-1}(s) \) has a pole, and \( G(s) \) a zero, in \( s = 1 \).

STABILITY

The characteristic behavior of the system response is determined by

- \( e^{At} \) in continuous-time
- \( A^T \) in discrete-time

\textit{Mathematical result:}

If the matrix \( A \) has eigenvalues \( p_1, \ldots, p_n \), the matrix exponential \( e^{At} \) has eigenvalues in \( e^{p_1 t}, \ldots, e^{p_n t} \).

For a \textit{continuous-time system} the stability region is the left half plane (without the imaginary axis).

For a \textit{discrete-time system} the stability region is the interior of the unit circle,
STABILITY cont’d

A linear time-invariant system is input-output stable \( \Leftrightarrow \) all its poles are located in the stability region.

A system with a zero outside the stability region is said to be non-minimum phase.

FREQUENCY FUNCTION

What does \( G(i\omega) \) tells us?

Assume the system is stable, and the input is

\[ u_k(t) = \cos(\omega t) \]

After a transient the output will be

\[ y_j(t) = A \cos(\omega t + \varphi) \quad j = 1, \ldots, p \]

with

\[ A = |G_{j,k}(i\omega)|, \quad \varphi = \arg [G_{j,k}(i\omega)] \]

(Use SISO reasoning and superposition).

Nontrivial how to cope with multivariable systems, and how to treat coupling between signals.

Can we relate the gain to an amplitude curve?

STABILITY, cont’d

Testing stability:

- Direct computation (in Matlab use \texttt{roots}, or \texttt{eig}).
- Root locus
- Nyquist criterion
- Routh (continuous-time) or Jury-Schur-Cohn (discrete-time)

FREQUENCY FUNCTIONS, cont’d

Singular value decomposition, SVD

Result: Every (rectangular) matrix \( A \) can be written as

\[ A = U \Sigma V^* \]

\( U, V \) unitary, \( \Sigma \) diagonal with positive elements

\[ \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ \vdots & \ddots \\ 0 & & \sigma_n \end{pmatrix} \]

\( \sigma_1, \ldots, \sigma_n \) singular values

\[ \| A \|_2 = \max_i \sigma_i = \sigma \]

\[ \sigma = \min_i \sigma_i \]

\[ y = Ax \implies \sigma |x| \leq |y| \leq \sigma |x| \]
FREQUENCY FUNCTIONS, cont’d

Make a SVD of $G(i\omega)$.

$$\sigma(G(i\omega)) \leq \frac{|Y(i\omega)|}{|U(i\omega)|} \leq \sigma(G(i\omega))$$

- The correspondence to the amplitude curve for SISO systems is a plot of the singular values of $G(i\omega)$ versus $\omega$.
- For multi-input systems the actual gain depends on the direction of the vector $U(i\omega)$.

EXAMPLE: A HEAT EXCHANGER

- $f_C$, $f_H$ flows
- $V_C$, $V_H$ volumes
- $T_C$, $T_H$ temperatures

Model

$$V_C \frac{dT_C}{dt} = f_C(T_C - T_C) + \beta(T_H - T_C)$$
$$V_H \frac{dT_H}{dt} = f_H(T_H - T_H) - \beta(T_H - T_C)$$

EXAMPLE HEAT EXCHANGER, cont’d

Simplified assumptions:

- $f_H = f_C = f$ is constant,
- Inputs: $u_1 = T_{C1}$, $u_2 = T_{H1}$
- States: $x_1 = T_C$, $x_2 = T_H$

State space model

$$\dot{x} = \begin{pmatrix}
-(f + \beta)/V_C & \beta/V_C \\
\beta/V_H & -(f + \beta)/V_H
\end{pmatrix} x + \begin{pmatrix} f/V_C & 0 \\
0 & f/V_H
\end{pmatrix} u$$

Numerical values: $f = 0.01$ (m$^3$/min),
$\beta = 0.2$ (m$^3$/min), $V_H = V_C = 1$ (m$^3$).

$$\dot{x} = \begin{pmatrix} -0.21 & 0.2 \\
0.2 & -0.21
\end{pmatrix} x + \begin{pmatrix} 0.01 & 0 \\
0 & 0.01
\end{pmatrix} u$$

HEAT EXCHANGER, cont’d

Transfer function matrix

$$G(s) = \begin{pmatrix} s + 0.21 & -0.2 \\
-0.2 & s + 0.21
\end{pmatrix}^{-1} \begin{pmatrix} 0.01 & 0 \\
0 & 0.01
\end{pmatrix} = \frac{0.01}{(s + 0.01)(s + 0.41)} \begin{pmatrix} s + 0.21 & 0.2 \\
0.2 & s + 0.21
\end{pmatrix}$$

USEFUL MATLAB COMMANDS

- `ss2zp`, `zp2ss`, `tf2zp`, `zp`.
  Transformations between representations
- `tzero`, Calculation of zero (for multivariable systems)
- `lsim`, Simulation
- `eig`, `roots`, Eigenvalues and poles
- `bode`, Frequency function, Bode plots
- `sigma`, Singular values of the transfer function
- `obsv`, `ctrb`, Observability, controllability