6. The closed loop system

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THE SET-UP (continuous or discrete time)

Open loop system
\[ z(t) = Gw(t) + w(t) \]
\[ y(t) = z(t) + n(t) \]
Feedback
\[ u(t) = F_r r(t) - F_y y(t) + w_u(t) \]
Disturbance model
\[ w(t) = G_d d(t) \]

THE CLOSED LOOP TRANSFER FUNCTIONS

Loop gain
\[ GF_y \text{ or } F_y G \]
Control error
\[ e(t) = r(t) - z(t) \]
Closed loop system
\[ z(t) = G_c r(t) + Sw(t) - Tn(t) + SGw_u(t) \]
\[ e(t) = (I - G_c) r(t) - Sw(t) + Tn(t) - SGw_u(t) \]
Input signal \( u(t) \):
\[ u(t) = S_u F_r r(t) - S_u F_y \left( w(t) + n(t) \right) + S_u w_u(t) \]

THE CLOSED LOOP TRANSFER FUNCTIONS, cont’d

Closed loop
\[ G_c = (I + GF_y)^{-1} GF_r \]
Sensitivity function
\[ S = (I + GF_y)^{-1} \]
Complementary sensitivity function
\[ T = (I + GF_y)^{-1} GF_y \]
Input sensitivity function
\[ S_u = (I + F_y G)^{-1} \]
Inputs: \( r(t), w(t), w_u(t), n(t) \)
Outputs: \( z(t) \) or \( e(t), u(t) \)
STABILITY

Take care of pole-zero cancellations!

Definition: The closed loop system is said to be internally stable if the following four transfer functions are stable (after possible cancellations), as well as $F_r$,

$$w_u(t) \mapsto u(t): S_u = (I + F_y G)^{-1}$$
$$w_u(t) \mapsto y(t): SG = (I + GF_y)^{-1}G$$
$$w(t) \mapsto u(t): -S_u F_y = -(I + F_y G)^{-1}F_y$$
$$w(t) \mapsto y(t): S = (I + GF_y)^{-1}$$

• If $F_y$ and $G$ are unstable, all four transfer functions must be checked.
• If $F_y$ is stable, it is enough to check
  $$G_{w_u \rightarrow y} = (I + GF_y)^{-1}G$$

STABILITY, cont’d

An equivalent criterion for internal stability

$$\begin{pmatrix}
    I & -G \\
    F_y & I
\end{pmatrix}^{-1}$$

is stable

Block scheme interpretation

Relations

$$\begin{pmatrix}
    I & -G \\
    F_y & I
\end{pmatrix}\begin{pmatrix}
    y \\
    u
\end{pmatrix} = \begin{pmatrix}
    w \\
    r
\end{pmatrix}$$

SENSITIVITY

Describe deviation between a true (unknown) system $G_0$ and a nominal model $G$.

Model description

$$G_0 = (I + \Delta_G)G$$

where $\Delta_G$ accounts for the uncertainties.

Neglect disturbances (think of superposition!). True output

$$z_0 = (I + G_0F_y)^{-1}G_0F_r r$$

to be compared with the nominal output

$$z = (I + GF_y)^{-1}GF_r r$$
**SENSITIVITY, cont’d**

Relation

\[ z_0 = (I + \Delta_z)z \]
\[ \Delta_z = S_0 \Delta_G \]
\[ S_0 = (I + G_0 F_y)^{-1} \]

The sensitivity function \( S_0 \) describes how the relative model error \( \Delta_G \) is transformed to a relative output error \( \Delta_z \).

**ROBUSTNESS**

*Question:* How large model errors can be accepted until stability is lost?

Assume \( \Delta_G \) is stable.

**ROBUSTNESS, cont’d**

*Equivalent model*

![Diagram of equivalent model](image)

Small gain theorem guarantees stability if

\[ \| \Delta_G \| \| T \| \| \infty < 1 \]

or

\[ |T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}, \quad \forall \omega \]

**DESIGN ASPECTS**

*Objectives:*

Choose a regulator, so that

- \( z(t) \) follows \( r(t) \),
- in the presence of disturbances, measurement errors, and modeling errors,
- with reasonable magnitude of the input signal.
DESIGN ASPECTS

Formalization

1. \(|I - G_c|\) must be small [\(z(t)\) to follow \(r(t)\)].
2. \(S\) must be small [small influence of process disturbances and modeling errors].
3. \(T\) must be small [small influence of measurement errors, guaranteed system stability].
4. \(G_{r+u}\) and \(G_{u+u}\) must not be too large [\(u\) of moderate magnitude].

*The requirements are in conflict! For example,*

\[
S + T = I \\
G_c = GG_{r+u}
\]

TIME DOMAIN SPECIFICATIONS

Design objectives 1
Choose the regulator so that
\(M\), \(e_0\), \(T_s\), \(T_r\) are smaller than given values.

TIME DOMAIN SPECIFICATIONS, cont’d

Error coefficients

\(e_0 = \lim_{t \to \infty} e(t) = I - G_c(0)\)

Assume \(F_y = F_r \Rightarrow I - G_c = I - T = S\).

Usual requirement (design)

\(e_0 = S(0) = 0\)

Ramp reference signal

\(e_1 = \lim_{t \to \infty} e(t) = \lim_{s \to 0} \frac{1}{s} \left[I - G_c(s)\right] = \frac{dS}{ds} |_{s=0}\)

TIME DOMAIN SPECIFICATIONS, cont’d

Design objectives 2
Choose the regulator so that
\(G_c\) and \(S\) are equal to given transfer functions.

Design objectives 3
Choose the regulator so that
\(G_c\) and \(S\) have a given set of poles.

Design objectives 4
Choose the regulator so that

\[\| e \|^2_{Q_1} + \| u \|^2_{Q_2}\]

is minimized.
FREQUENCY DOMAIN SPECIFICATIONS

\[ |S(i\omega)| \leq |W_S^{-1}(i\omega)|, \quad \forall \omega \]
\[ |T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}, \quad \forall \omega \]

In the multivariable case
\[ \| W_S S \|_\infty \leq 1 \]
\[ \| W_T T \|_\infty < 1 \]

Design objectives 5
Choose the regulator so that
\[ \| W_S S \|_\infty < 1 \]
\[ \| W_T T \|_\infty < 1 \]
\[ \| W_u G_{ru} \|_\infty < 1 \]

Design objectives 6
Choose the regulator so that the criterion
\[ \| W_S S \|_2^2 + \| W_T T \|_2^2 + \| W_u G_{ru} \|_2^2 \]
is minimized.