8. Regulator structures

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A control system with two degrees of freedom

\[ r \rightarrow F_r \rightarrow r \rightarrow G \rightarrow y \]

An equivalent form:

\[ r \rightarrow \hat{r} \rightarrow F_r \rightarrow F_y \rightarrow u \rightarrow G \rightarrow y \]

APPROACHES FOR CONTROL LOOP DESIGN

1. Signal based design. Characterize performance in terms of signal properties. Express design objectives with a criterion to be optimized. See LQG, MPC (Chapters 9 and 16)

2. Loop based design. Characterize performance in terms of system properties (loop gain, sensitivity, robustness, etc). See Loop shaping (Chapter 10)

DESIGN PROCESS

1. Modeling (incl modeling of disturbances).
2. Specification considerations. Check design objectives versus inherent limitations.
3. Design. Choose a regulator structure, and select its parameters.
4. Test and validation. Use simulations to examine if all design objectives are satisfied.
**RELATIVE GAIN MATRIX**

Coupling between different loops can be difficult!

RGA (Relative Gain Array) is a measure of the cross-couplings in a matrix:

\[ \text{RGA}(A) = A \odot (A^{-1})^T \]

\( \odot \) denotes element-wise multiplication

Generalized definition – rectangular case:

\[ \text{RGA}(A) = A \odot (A^H)^T \]

where \( A^+ \) is the pseudo-inverse.

**RELATIVE GAIN MATRIX, cont’d**

Some properties of RGA:

1. All row sums and all column sums of RGA have magnitude equal to 1.
2. RGA is independent of scaling:
   \[ \text{RGA}(D_1 A D_2) = \text{RGA}(A) \]
3. \( \sum_{i,j} |\text{RGA}(A)|_{i,j} \) is a good measure of \( \text{cond}(A) \).
4. Permutation of rows [columns] of \( A \) leads to the same permutation of rows [columns] of \( \text{RGA}(A) \).
5. If \( A \) is diagonal, \( \text{RGA}(A) = I \).

The deviation of \( \text{RGA}(A) \) from \( I \) is a measure of the cross-couplings.

**THE PAIRING PROBLEM**

*Idea:* Use decentralized control. Let every input be determined by feedback from one single output. (If \# of inputs \# outputs, neglect the superfluous ones.)

The *pairing problem* is to select which input-output pairs that should be used for the feedback.

**Design rules**

1. Select input-output pairs so that the diagonal elements of \( \text{RGA}(G(i\omega_c)) \) are close to 1; \( \omega_c \) = designed bandwidth.
2. Avoid pairing that gives negative diagonal elements of \( \text{RGA}(G(0)) \).

**DECOUPLED CONTROL**

If it is difficult to find decentralized controllers, try to use filtered signals

\[ \tilde{u} = W_1^{-1} u, \quad \tilde{y} = W_2 y \]

so that the transfer function from \( \tilde{u} \) to \( \tilde{y} \)

\[ \tilde{G}(s) = W_2(s)G(s)W_1(s) \]

is close to diagonal. Apply decentralized control to \( \tilde{u}, \tilde{y} \) and transform back to \( u, y \).

In practice, treat the case \( s = 0 \) (stationarity) or \( s = i\omega_c \). Possibly \( W_1(s) = G^{-1}(0) \), \( W_2(s) = I \) may work [\( \tilde{G}(0) = I \)].
INTERNAL MODEL CONTROL

![Block diagram of internal model control system]

True system: \( G_0 \), Model: \( G \).

Design feedback from \( y - Gu \) (compensate for disturbances, measurement errors, modeling errors)

\[
 u = -Q(y - Gu)
\]

\( Q \) can be seen as a way of describing or parameterizing the feedback.

INTERNAL MODEL CONTROL, cont’d

Feedback

\[
 u = -Q(y - Gu) + Q\bar{r}, \quad \bar{r} = \bar{F}_r r
\]

Feedback from \( y \):

\[
 F_y = (I - QG)^{-1}Q
\]

Closed-loop system in the nominal case \( (G_0 = G) \):

\[
 G_c = (I + GF_y)^{-1}GF_y \bar{F}_r = GQ\bar{F}_r
\]

INTERNAL MODEL CONTROL, cont’d

Complementary sensitivity function

\[
 T = (I + GF_y)^{-1}GF_y = GQ
\]

Sensitivity function

\[
 S = I - T = I - GQ
\]

Note, both \( S \) and \( T \) are linear in \( Q \).

In case of modeling error \( (G_0 \neq G) \)

\[
 G_c = G_0 \left( I + Q(G_0 - G) \right)^{-1} Q\bar{F}_r
\]

INTERNAL MODEL CONTROL, cont’d

Assume the open loop system \( G \) is stable, and that there are no modeling errors \( (G_0 = G) \).

All the relevant transfer functions are stable if \( Q \) is stable.

The set of all stabilizing controllers is given by

\[
 u = -(I - QG)^{-1}Qy
\]

with \( Q \) stable (Youla-Kucera parameterization).

All the transfer functions are linear in \( Q \).

How should \( Q \) be designed?


**INTERNAL MODEL CONTROL, cont’d**

_Design of Q_

An ‘ideal’ case: \( Q = G^{-1} \) would give 
\( S = 0, \ G_c = I, \) but is infeasible as \( F_y = \infty. \)

_Some rules of thumb_

1. If \( G(s) \) is strictly proper, take 
\[
Q(s) = \frac{1}{(\lambda s + 1)^n} G^{-1}(s)
\]

Choose \( n \) so that \( Q(s) \) is proper.
Choose \( \lambda \) to adjust the bandwidth of the closed loop system.

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**INTERNAL MODEL CONTROL, cont’d**

2. If \( G(s) \) has an unstable zero, and contains a factor \((s - \beta), \beta > 0:\)

(a) Omit the factor when \( Q = G^{-1} \) is formed.

(b) Replace the factor with \((s + \beta)\) when \( Q = G^{-1} \) is formed.

3. If \( G(s) \) contains a time-delay, hence a factor \( e^{-\tau s}. \)

(a) Neglect the factor when forming \( Q. \)

(b) Approximate the factor with 
\[
e^{-\tau s} \approx \frac{1 - s \tau/2}{1 + s \tau/2}
\]

and use methods for Case 2.

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**λ TUNING**

Set 
\[
Q(s) = \frac{1}{(\lambda s + 1)^n} G^{-1}(s)
\]

Then

\[
F_y(s) = \frac{1}{(\lambda s + 1)^n - 1} G^{-1}(s)
\]
\[
S(s) = \left( 1 - \frac{1}{(\lambda s + 1)^n} \right) I
\]
\[
T(s) = \frac{1}{(\lambda s + 1)^n} I
\]
\[
G_c(s) = \frac{1}{(\lambda s + 1)^n} F_r(s)
\]

bandwidth \( \approx 1/\lambda. \)

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**EXAMPLE λ TUNING**

A DC servo

\[
G(s) = \frac{1}{s(s + 1)^n}, \quad n = 2
\]

\[
F_y(s) = \frac{s + 1}{\lambda(\lambda s + 2)}
\]
\[
S(s) = \frac{\lambda^2 s^2 + 2\lambda s}{(\lambda s + 1)^2}
\]
\[
T(s) = \frac{1}{(\lambda s + 1)^2}
\]
\[
G_c(s) = \frac{1}{(\lambda s + 1)^2} F_r
\]
TUNING, cont’d

Closed loop system, general case

\[
\begin{pmatrix}
  z \\
u
\end{pmatrix} = \begin{pmatrix}
  G_c & S & -T \\
  S_u F_r & -S_u F_y & -S_u F_y
\end{pmatrix} \begin{pmatrix}
  r \\
w
\end{pmatrix}
\]

where

\[
G_c(s) = \frac{1}{(\lambda s + 1)^n} F_r(s)
\]

\[
S(s) = \left(1 - \frac{1}{(\lambda s + 1)^n}\right) I
\]

\[
T(s) = \frac{1}{(\lambda s + 1)^n} I
\]

\[
S_u F_y = \frac{1}{(\lambda s + 1)^n} G^{-1}(s)
\]

\[
S_u F_r = \frac{1}{(\lambda s + 1)^n} G^{-1}(s) F_r(s)
\]

FEEDBACK FROM ESTIMATED STATES

Open loop system

\[
px = Ax + Bu + N v_1
\]

\[
z = M x
\]

\[
y = C x + v_2
\]

Feedback

\[
p\dot{x} = Ax + Bu + K(y - C\dot{x})
\]

\[
u = -L\dot{x} + \hat{r}
\]

The feedback is parametrized with the matrices $L$ ($m \times n$) and $K$ ($n \times p$).

FEEDBACK FROM ESTIMATED STATES, cont’d

Compute transfer functions!

Feedback

\[
u = -F_y(p)y + F_r(p)\hat{r}
\]

\[
F_y(p) = L(pI - A + BL + KC)^{-1} K
\]

\[
F_r(p) = I - L(pI - A + BL + KC)^{-1} B
\]

Loop gain

\[
G(p)F_y(p) = C(pI - A)^{-1}BF_y(p)
\]

FEEDBACK FROM ESTIMATED STATES, cont’d

The closed loop system.

Use state vector

\[
X = \begin{pmatrix}
x \\
x - \hat{x}
\end{pmatrix}
\]

\[
pX = \begin{pmatrix}
A - BL & BL \\
0 & A - KC
\end{pmatrix} X + \begin{pmatrix}
B \\
0
\end{pmatrix} \hat{r}
\]

\[
+ \begin{pmatrix}
N \\
N
\end{pmatrix} v_1 + \begin{pmatrix}
0 \\
-K
\end{pmatrix} v_2
\]

\[
z = \begin{pmatrix}
M \\
0
\end{pmatrix} X
\]

Closed loop poles: eigenvalues of $A - BL$ and eigenvalues of $A - KC$.  

INTRODUCING INTEGRATING REGULATORS

Avoid stationary control errors by requiring

\[ S(0) = 0 \]

The open loop system or the regulator must include an integrator.

\[ \begin{align*}
    w &\rightarrow r \\
    \Sigma &\rightarrow u \\
    G &\rightarrow y \\
    -F_y &\rightarrow y
\end{align*} \]

Example (oscillator) \[ G(s) = \frac{\omega_0^2}{s^2 + \omega_0^2} \]

\[
\begin{align*}
    \dot{x} &= \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \omega_0^2 \end{pmatrix} u \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x
\end{align*}
\]

Use pole placement

Closed loop poles chosen as

\[-\omega_0 \pm i\omega_0\]

INTEGRATING REGULATORS, cont’d

Disturbance \( w \) a step at time \( t = 5 \).

State feedback

\[
\begin{align*}
    x_n^\omega &\rightarrow x_n^\omega \\
    x_n^b &\rightarrow x_n^b \\
    x_n^b &\rightarrow x_n^b
\end{align*}
\]

Feedback from estimated states

Observer poles chosen as \( 1.2(\pm \omega_0 \pm i\omega_0) \)
INTEGRATING CONTROLLERS, cont’d

Extended model

State vector
\[ \mathbf{x} = \begin{pmatrix} x \\ w \end{pmatrix} \]

Model
\[ \dot{\mathbf{x}} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} B \\ 0 \end{pmatrix} u \]
\[ y = \begin{pmatrix} C & 0 \end{pmatrix} \mathbf{x} \]

Feedback
\[ u = -L \dot{x} - L_w \dot{\omega} \]
\[ \frac{d}{dt} \begin{pmatrix} \dot{x} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\omega} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} K \\ K_1 \end{pmatrix} (y - C \dot{x}) \]

INTEGRATING REGULATORS, cont’d

Closed loop system
\[ \dot{\tilde{\mathbf{x}}} = \begin{pmatrix} A - KC & B \\ -K_1 C & 0 \end{pmatrix} \tilde{\mathbf{x}} \]
\[ \dot{\tilde{\mathbf{x}}} = \begin{pmatrix} A - BL & B - BL_w \\ 0 & 0 \end{pmatrix} \tilde{\mathbf{x}} + \begin{pmatrix} B \\ 0 \end{pmatrix} \begin{pmatrix} L & L_w \end{pmatrix} \tilde{\mathbf{x}} \]

User parameters:
\[ L_w = 1 \Rightarrow B - BL_w = 0, \quad K_1 = ? \]

Simulation result for integrating regulator

Tuned parameter: \( K_1 = 3 \).