Control design, Dec 14, 2001 — Answers and brief solutions

Problem 1

a) The general rule is that an increased \( R_1 \) decreases \( S \). Similarly, an increased \( R_2 \) decreases \( T \).

Hence the result is

- case 1 – curves A, c
- case 2 – curves C, b
- case 3 – curves B, a

b) The Kalman gain is independent of a rescaling of the covariance matrices. If \( R_1 \) and \( R_2 \) gives a solution \( P \) to the Riccati equation and a Kalman gain \( K \), then the penalty matrices \( \beta R_1 \) and \( \beta R_2 \) gives a solution \( \beta P \) and the same Kalman gain. The regulator will hence be the same as in case 1, and the sensitivity functions will hence of course be identical to those of case 1.

Problem 2

(a) The transfer function becomes

\[
G(s) = (sI - A)^{-1} = \frac{1}{s^2 + 2(1 + \alpha)s + 1 + 2\alpha} \begin{pmatrix} s + 1 + \alpha & \alpha \\ \alpha & s + 1 + \alpha \end{pmatrix}
\]

(b) The poles of the system are the eigenvalues of the matrix \( A \), that is the solutions to

\[
0 = \det (sI - A) = (s + 1 + \alpha)^2 - \alpha^2 = (s + 1)(s + 1 + 2\alpha)
\]

One eigenvalue is hence always in \( s = -1 \), while the other is in \( s = -1 - 2\alpha \).

(c) The zeros of the system can be found as the poles of \( G^{-1}(s) \) (as the system has an equal number of inputs and outputs). Noting that

\[
G^{-1}(s) = sI - A = \begin{pmatrix} s + 1 + \alpha & -\alpha \\ -\alpha & s + 1 + \alpha \end{pmatrix}
\]

which has no poles, we note that the system has no zeros.
Problem 3

(a) For $\alpha = 0.5$ one gets

$$G(0) = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$$

leading to

$$RGA(G(0)) = \begin{pmatrix} 1.125 & -0.125 \\ -0.125 & 1.125 \end{pmatrix}$$

The second alternative (to make the pairing $u_1 - y_2$ and $u_2 - y_1$) will hence give nonnegative elements in RGA, and should therefore be avoided.

(b) The requirement on static decoupling gives

$$W_1 = G^{-1}(0) = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{pmatrix}$$

(c) The open loop system is

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

and the regulator is

$$u = W_1 K (r - y)$$

with $W_1 = I$ when there is no decoupling, and $K = 10I$. The closed loop system is

$$\dot{x} = Ax + BW_1 K (r - Cx)$$
$$= (A - BW_1 KC) x + BW_1 Kr$$
$$= (A - W_1 K) x + W_1 Kr$$

where we have used that $B = I$, $C = I$. The closed loop poles are the eigenvalues of $A - W_1 K$.

For the case of no decoupling,

$$A - W_1 K = \begin{pmatrix} -1.5 & 0.5 \\ 0.5 & -1.5 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} -11.5 & 0.5 \\ 0.5 & -11.5 \end{pmatrix}$$

which has eigenvalues in $-11.5 \pm 0.5$, thus is in $s = -11$, $s = -12$.

For the case of decoupling,

$$A - W_1 K = \begin{pmatrix} -1.5 & 0.5 \\ 0.5 & -1.5 \end{pmatrix} - \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{pmatrix} \times 10 \times I = \begin{pmatrix} -16.5 & 5.5 \\ 5.5 & -16.5 \end{pmatrix}$$

which has eigenvalues in $-16.5 \pm 5.5$, thus is in $s = -11$, $s = -22$. 

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Problem 4

The transfer function $Q(s)$ becomes

$$Q(s) = \frac{1}{(\lambda s + 1)^2} G^{-1}(s) = \frac{1}{(\lambda s + 1)^2} \frac{s^2 + 2\zeta \omega_n s + \omega_n^2}{\omega_n^2}$$

and the regulator will be

$$F_y(s) = [1 - Q(s)G(s)]^{-1}Q(s) = \frac{1}{1 - \frac{1}{(\lambda s + 1)^2}} \frac{s^2 + 2\zeta \omega_n s + \omega_n^2}{\omega_n^2}$$

As $F_y(0) = \infty$, the regulator is integrating.

Problem 5

a) Introduce the new variables as

$$z_1 = W_u u = bu$$
$$z_2 = W_y y = \frac{a}{p} y$$
$$z_3 = W_T x_1 = 0.5 x_1$$

Only $z_2$ involves some new dynamics. Introduce the additional state variable

$$x_2 = z_2 = \frac{a}{p} (x_1 + w)$$

so $\dot{x}_2 = ax_1 + aw$. The state space model for the extended system becomes

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ a \end{bmatrix} w$$

$$y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x + w$$

$$z = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0.5 & 0 \end{bmatrix} x + \begin{bmatrix} b \\ 0 \end{bmatrix} u$$

b) It holds that

$$D^T M = 0, \ D^T D = b^2$$

In case $b^2 \neq 1$, rescale the problem. Introduce

$$\bar{u} = bu$$
as the scaled input. The rewritten model becomes
\[
\begin{align*}
\dot{x} &= \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} x + \begin{pmatrix} 1/b \\ 0 \end{pmatrix} w + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w \\
y &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} x + w \\
z &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w
\end{align*}
\]

**Problem 6**

To treat the general case, let the measurement noise have intensity \( r_2 \). The estimation error \( \tilde{x} \) satisfies
\[
\dot{\tilde{x}} = (-1 - K) \tilde{x} - Ke + v
\]

Applying the Lyapunov equation then gives easily
\[
V(K) = \frac{K^2 r_2 + 1}{2(1 + K)}
\]

(a) Setting \( r_2 = 1 \) gives
\[
V(K) = \frac{K^2 + 1}{2(1 + K)}
\]

(b) The minimizing element of \( V(K) \) is found to be \( K^* = -1 + \sqrt{2} \). Further, the minimal value turns out to be \( V = \sqrt{2} - 1 \).

(c) The associated Riccati equation is
\[
0 = -P - P + 1 - P^2 \times 1^2 / r_2
\]

which leads to
\[
P^2 + 2r_2 P - r_2 = 0
\]

with the solution \( P = -r_2 \pm \sqrt{r_2^2 + r_2} \). As \( r_2 = 1 \) in part (c), the positive solution is \( P = \sqrt{2} - 1 \), as \( V \) in part (b).

(d) The variance using the fixed observer gain \( K^* \) for the noise intensity \( r_2 = 1/3 \) becomes
\[
V = \frac{(K^*)^2/3 + 1}{2(1 + K^*)} = \frac{1}{2} - \frac{1}{3} \approx 0.374
\]

(e) The minimal variance of the estimation error, when \( r_2 = 1/3 \), is given by the solution to the Riccati equation
\[
0 = -P - P + 1 - P^2 / (1/3)
\]

which is \( P = 1/3 \approx 0.333 \).