Computer controlled systems, Dec 14, 2001 — Answers and brief solutions

Problem 1

(a) The transfer function becomes

\[ G(s) = (sI - A)^{-1} = \frac{1}{s^2 + 2(1 + \alpha)s + 1 + 2\alpha} \begin{pmatrix} s + 1 + \alpha & \alpha \\ \alpha & s + 1 + \alpha \end{pmatrix} \]

(b) The poles of the system are the eigenvalues of the matrix \( A \), that is the solutions to

\[ 0 = \det(sI - A) = (s + 1 + \alpha)^2 - \alpha^2 = (s + 1)(s + 1 + 2\alpha) \]

One eigenvalue is hence always in \( s = -1 \), while the other is in \( s = -1 - 2\alpha \).

(c) The zeros of the system can be found as the poles of \( G^{-1}(s) \) (as the system has an equal number of inputs and outputs). Noting that

\[ G^{-1}(s) = sI - A = \begin{pmatrix} s + 1 + \alpha & -\alpha \\ -\alpha & s + 1 + \alpha \end{pmatrix} \]

which has no poles, we note that the system has no zeros.

Problem 2

(a) For \( \alpha = 0.5 \) one gets

\[ G(0) = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix} \]

leading to

\[ RGA(G(0)) = \begin{pmatrix} 1.125 & -0.125 \\ -0.125 & 1.125 \end{pmatrix} \]

The second alternative (to make the pairing \( u_1 - y_2 \) and \( u_2 - y_1 \)) will hence give nonnegative elements in RGA, and should therefore be avoided.

(b) The requirement on static decoupling gives

\[ W_1 = G^{-1}(0) = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{pmatrix} \]
(c) The open loop system is

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

and the regulator is

\[ u = W_1 f(r - y) \]

with \( W_1 = I \) when there is no decoupling, and \( K = 10I \). The closed loop system is

\[ \dot{x} = Ax + BW_1 K(r - Cx) \]
\[ = (A - BW_1 KC)x + BW_1 Kr \]
\[ = (A - W_1 K)x + W_1 Kr \]

where we have used that \( B = I, C = I \). The closed loop poles are the eigenvalues of \( A - W_1 K \).

For the case of no decoupling,

\[ A - W_1 K = \begin{pmatrix} -1.5 & 0.5 \\ 0.5 & -1.5 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} -11.5 & 0.5 \\ 0.5 & -11.5 \end{pmatrix} \]

which has eigenvalues in \(-11.5 \pm 0.5\), that is, in \( s = -11, s = -12 \).

For the case of decoupling,

\[ A - W_1 K = \begin{pmatrix} -1.5 & 0.5 \\ 0.5 & -1.5 \end{pmatrix} - \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{pmatrix} \times 10 \times I = \begin{pmatrix} -16.5 & 5.5 \\ 5.5 & -16.5 \end{pmatrix} \]

which has eigenvalues in \(-16.5 \pm 5.5\), that is, in \( s = -11, s = -22 \).

**Problem 3**

The transfer function \( Q(s) \) becomes

\[ Q(s) = \frac{1}{(\lambda s + 1)^2} G^{-1}(s) = \frac{1}{(\lambda s + 1)^2} \frac{s^2 + 2\zeta \omega_0 s + \omega_0^2}{\omega_0^2} \]

and the regulator will be

\[ F_y(s) = [1 - Q(s)G(s)]^{-1}Q(s) \]
\[ = \frac{1}{1 - \frac{1}{(\lambda s + 1)^2}} \frac{s^2 + 2\zeta \omega_0 s + \omega_0^2}{(\lambda s + 1)^2 \omega_0^2} \]
\[ = \frac{1}{\lambda^2 s^2 + 2\lambda s} \frac{s^2 + 2\zeta \omega_0 s + \omega_0^2}{\omega_0^2} \]

As \( F_y(0) = \infty \), the regulator is integrating.
Problem 4

a) The general rule is that an increased \( R_1 \) decreases \( S \). Similarly, an increased \( R_2 \) decreases \( T \).

Hence the result is

- case 1 – curves A, c
- case 2 – curves C, b
- case 3 – curves B, a

b) The Kalman gain is independent of a rescaling of the covariance matrices. If \( R_1 \) and \( R_2 \) gives a solution \( P \) to the Riccati equation and a Kalman gain \( K \), then the penalty matrices \( \beta R_1 \) and \( \beta R_2 \) gives a solution \( \beta P \) and the same Kalman gain. The regulator will hence be the same as in case 1, and the sensitivity functions will hence of course be identical to those of case 1.

Problem 5

a) Introduce the new variables as

\[
\begin{align*}
  z_1 &= W_1 u = bu \\
  z_2 &= W_2 y = \frac{a}{p} y \\
  z_3 &= W_3 x_1 = 0.5 x_1
\end{align*}
\]

Only \( z_2 \) involves some new dynamics. Introduce the additional state variable

\[
x_2 = z_2 = \frac{a}{p} (x_1 + w)
\]

so \( \dot{x}_2 = a x_1 + aw \). The state space model for the extended system becomes

\[
\begin{align*}
  \dot{x} &= \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ a \end{pmatrix} w \\
  y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + w \\
  z &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0.5 & 0 \end{pmatrix} x + \begin{pmatrix} b \\ 0 \end{pmatrix} u
\end{align*}
\]

b) It holds that

\[
D^T M = 0, \quad D^T D = b^2
\]

In case \( b^2 \neq 1 \), rescale the problem. Introduce

\[
\bar{u} = bu
\]
as the scaled input. The rewritten model becomes
\[
\begin{align*}
\dot{x} &= \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} x + \begin{pmatrix} 1/b \\ 0 \end{pmatrix} \pi + \begin{pmatrix} 0 \\ a \end{pmatrix} w \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + w \\
z &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0.5 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ \pi \end{pmatrix}
\end{align*}
\]

**Problem 6**

(a) There are three equilibrium points
\[
\begin{align*}
x &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} \sqrt{6} \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} -\sqrt{6} \\ 0 \end{pmatrix}
\end{align*}
\]
The Jacobian is
\[
\frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ -1 + x_1^2/2 & -1 \end{pmatrix}
\]
For the first point, we have
\[
A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}
\]
with eigenvalues in \( s = -0.5 \pm i\sqrt{0.75} \). The equilibrium is a stable focus.

For the second point, we have
\[
A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}
\]
with eigenvalues in \( s = 1 \) and \( s = -2 \). The equilibrium is a saddle point.

For the third point, we have
\[
A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}
\]
with eigenvalues in \( s = 1 \) and \( s = -2 \). The equilibrium is a saddle point.

(b) There is a unique stationary point, \( x = (0 \ 0)^T \). The Jacobian becomes
\[
A = \frac{\partial f}{\partial x} \bigg|_{x=0} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}
\]
As it has eigenvalues in \( s = 0.5 \pm i\sqrt{0.75} \), it is an unstable focus.
(c) From the second equation we find that either \( x_2 = 0 \) or \( x_2 = 2(1 + x_1) \). Inserting this into the first equation, we find that there are in total four equilibrium points, as described below. Further, the Jacobian is in the general case

\[
\frac{\partial f}{\partial x} = \begin{pmatrix}
1 - 2x_1 - \frac{2x_2}{(1 + x_1)^2} & -2 \frac{x_1}{(1 + x_1)} \\
\frac{x_2^2}{(1 + x_1)^2} & 2 - \frac{2x_2}{(1 + x_1)}
\end{pmatrix}
\]

The first stationary point \( x = (0 \ 0)^T \) gives

\[
A = \begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix}
\]

It has eigenvalues in \( s = 1 \) and \( s = 2 \), and is an unstable focus.

The second stationary point \( x = (1 \ 0)^T \) gives

\[
A = \begin{pmatrix}
-1 & -1 \\
0 & 2
\end{pmatrix}
\]

It has eigenvalues in \( s = -1 \) and \( s = 2 \), and is a saddle point.

The third stationary point \( x = (0 \ 2)^T \) gives

\[
A = \begin{pmatrix}
-3 & 0 \\
4 & -2
\end{pmatrix}
\]

It has eigenvalues in \( s = -3 \) and \( s = -2 \), and is a stable node.

The fourth stationary point \( x = (-3 \ -4)^T \) gives

\[
A = \begin{pmatrix}
9 & -3 \\
4 & -2
\end{pmatrix}
\]

It has eigenvalues in \( s = 7.772 \) and \( s = -0.772 \), and is a saddle point.