Control Design, Dec 16, 2003 — Answers and brief solutions

Problem 1

(a) Determine first the pole polynomial. The $1 \times 1$ minors are the matrix elements. It is enough to consider

$$\frac{2}{s+1}, \quad \frac{3}{s+2}$$

There are 3 different $2 \times 2$ minors (each obtained by deleting one column of $G(s)$ when forming the determinant). These minors are

$$\frac{2}{s+1} \times \frac{1}{s+1} - \frac{3}{s+2} \times \frac{1}{s+1} = \frac{2(s+2) - 3(s+1)}{(s+1)^2(s+2)} = \frac{(-s+1)}{(s+1)^2(s+2)},$$

$$\frac{2}{s+1} \times \frac{1}{s+1} - \frac{3}{s+2} \times \frac{1}{s+1} = \frac{2(s+2) - 3(s+1)}{(s+1)^2(s+2)} = \frac{(-s+1)}{(s+1)^2(s+2)},$$

$$\frac{3}{s+2} \times \frac{1}{s+1} - \frac{3}{s+2} \times \frac{1}{s+1} = 0$$

The least common denominator for all the minors, that is the pole polynomial, is hence

$$(s+1)^2(s+2)$$

The system has a double pole in $s = -1$ and a single pole in $s = -2$.

To find the zeros of the system, consider the numerators of the $2 \times 2$ minors. These minors have already the pole polynomial as denominator. The zero polynomial is therefore $-s+1$, and the system has one zero in $s = 1$.

(b) Set

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

We then get the transfer function

$$G(s) = C(sI - A)^{-1}B$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s+1 & 0 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{s+1} & 0 & \frac{1}{s+2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$
\[
= \left( \begin{array}{ccc}
\frac{b_{11}}{s+1} + \frac{b_{13}}{s+2} & \frac{b_{12}}{s+1} + \frac{b_{13}}{s+2} & \frac{b_{13}}{s+1} + \frac{b_{13}}{s+2} \\
\frac{b_{23}}{s+1} & \frac{b_{23}}{s+1} & \frac{b_{33}}{s+1} \\
\frac{b_{33}}{s+1} & \frac{b_{33}}{s+1} & \frac{b_{33}}{s+1}
\end{array} \right)
\]

Comparing with the given expression for \( G(s) \) we find that
\[
B = \left( \begin{array}{ccc}
2 & 0 & 0 \\
1 & 1 & 1 \\
0 & 3 & 3
\end{array} \right)
\]

**Problem 2**

The Riccati equations give
\[\begin{align*}
-Z^2 + 1 &= 0 \\
Z &= 1 \\
-X^2 + 1 &= 0 \\
X &= 1
\end{align*}\]

Hence
\[\lambda = 1\]

As \( \gamma = \alpha \sqrt{2} \) one further gets
\[\begin{align*}
R &= 1 - \frac{1}{\alpha^2 2} = \frac{\alpha^2 - 1}{\alpha^2} \\
L &= 1 \\
K &= \frac{\alpha^2}{\alpha^2 - 1}
\end{align*}\]

Hence the feedback becomes
\[
F_y(s) = \frac{LK}{s - A + BL + KC} = \frac{LK}{s + L + K}
\]

\[= \frac{\alpha^2}{s + \frac{\alpha^2 - 1}{\alpha^2 - 1}}\]

When \( \alpha \to 1 \), \( R \) will become singular, and \( K \) will not exist. The limit of the feedback is a proportional regulator of gain
\[
\lim_{\alpha \to 1} F_y(s) = \lim_{\alpha \to 1} \frac{\alpha^2}{2\alpha^2 - 1} = 1
\]

**Problem 3**

The Nyquist diagram shows \( L(i\omega) \) in the complex plane with \( \omega \) as a parameter.

(a) The condition \( |L(i\omega_c)| = 1 \) is a circle with centre in the origin and with radius 1.
(b) As the sensitivity function is \( S(s) = 1/[1 + L(s)] \), the condition \( |S(i\omega_o)| = 1 \) becomes

\[
|1 + L(i\omega_o)| = 1
\]

It defines a circle with center in \( s = -1 \) and with radius 1.

(c) Noting that \( G_c(s) = L(s)/(1 + L(s)) \), the condition for the bandwidth \( \omega_B \) can be written as

\[
\left| \frac{L(i\omega_B)}{1 + L(i\omega_B)} \right| = \frac{1}{\sqrt{2}}
\]

Set

\[
L(i\omega_B) = a + ib
\]

Then

\[
\sqrt{2}|a + ib| = |1 + a + ib|
\]

or

\[
2(a^2 + b^2) = (1 + a)^2 + b^2
\]

which can be simplified to

\[
(a - 1)^2 + b^2 = 2
\]

This describes a circle with centre at \( s = 1 \) and with radius \( \sqrt{2} \).

See Figure 1 for illustration of the circles.

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![Figure 1: Nyquist curve for the loop gain \( L(i\omega) \), and intersections with the circles in Problem 4, parts (a) - (c).](image-url)
Problem 4

(a) A static gain equal to one from $r$ to $y$ gives

$$
m = \frac{1}{C(A - BL)^{-1}B}$$

$$= \frac{-1}{(1 + \ell_1)}\frac{1}{(1 - \ell_1 - \ell_2)}^{-1}\left(\begin{array}{c}
0 \\
1
\end{array}\right)$$

$$= \frac{-1}{(1 + \ell_1)}\left(\begin{array}{c}
-\ell_2 \\
1 + \ell_1 - 1
\end{array}\right)^{-1}\left(\begin{array}{c}
0 \\
1
\end{array}\right) = (1 + \ell_1)$$

(b) One gets directly from (a)

$$|1 + \ell_1| \leq u_o$$

(c) Denote the solution to the Riccati equation

$$0 = A^T S + SA + Q_1 - SBQ_2^{-1}B^T S$$

by

$$S = \left(\begin{array}{cc}
s_{11} & s_{12} \\
s_{12} & s_{22}
\end{array}\right)$$

Then the feedback vector $L$ is

$$\left(\begin{array}{c}
\ell_1 \\
\ell_2
\end{array}\right) = Q_2^{-1}B^T S = \left(\begin{array}{cc}
0 & 1
\end{array}\right)\left(\begin{array}{cc}
s_{11} & s_{12} \\
s_{12} & s_{22}
\end{array}\right) = \left(\begin{array}{c}
s_{12} \\
s_{22}
\end{array}\right)$$

Hence, the design constraint becomes

$$|1 + s_{12}| \leq u_o$$

Evaluating both sides of the Riccati equation leads to the nonlinear system of equations

$$0 = -2s_{12} + \rho - s_{12}^2$$

$$0 = -s_{22} + s_{11} - s_{12}s_{22}$$

$$0 = 2s_{12} - s_{22}^2$$

Here the first equation gives $s_{12}$, then the third gives $s_{22}$, and finally $s_{11}$ can be found from the second one. For this problem, though, it is sufficient to solve the first equation to get

$$s_{12} = -1 \pm \sqrt{1 + \rho}$$

From the third equation we find that $s_{12}$ must be positive. Hence we get

$$s_{12} = -1 + \sqrt{1 + \rho}$$

and

$$|m| \leq u_o \Rightarrow |1 + s_{12}| \leq u_o \Rightarrow \sqrt{1 + \rho} \leq u_o \Rightarrow \rho \leq u_o^2 - 1$$
Problem 5

(a)

\[ F_y(s) = [1 - Q(s)G(s)]^{-1}Q(s) = \frac{s + 1}{(\lambda s + 1)^k} 1 - \frac{s + 1}{(\lambda s + 1)^{k+1} s + 1} \]

with \( k = 1 \) for Civ and \( k = 2 \) for Civerth. Both regulators will be integrating. More explicitly,

Civ’s case: \( F_y(s) = \frac{s + 1}{\lambda s} \).

Civerth’s case: \( F_y(s) = \frac{s + 1}{\lambda^2 s^2 + 2\lambda s} = \frac{s + 1}{\lambda(\lambda s + 2)} \).

(b)

\[ Y(s) = G(s)Q(s)R(s) = \frac{1}{(\lambda s + 1)^k} R(s) \]

Problem 6

(a) The Riccati equation reads here

\[ 0 = \beta^2 r - P^2/r \implies P = \beta r \]

and the Kalman gain is \( K = P/r = \beta \). The Kalman filter is

\[ \dot{x} = K(y - \hat{x}) \implies \dot{x} = \frac{K}{p + K} y = \frac{\beta}{p + \beta} y \]

The error variance is \( E\tilde{x}^2(t) = P = \beta r \).

(b) The Riccati equation becomes

\[ 0 = -2\alpha P + \beta^2 r - P^2/r \implies P = -\alpha r + \sqrt{\alpha^2 + \beta^2} r \]

The Kalman gain is

\[ K = P/r = -\alpha + \sqrt{\alpha^2 + \beta^2} \]

The Kalman filter is

\[ \dot{x} = -\alpha \dot{x} + K(y - \hat{x}) \implies \]

\[ \dot{x} = \frac{K}{p + \alpha + K} y = \frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{p + \sqrt{\alpha^2 + \beta^2}} y \]

The error variance is

\[ P = \left(-\alpha + \sqrt{\alpha^2 + \beta^2}\right) r \]
(c) The process and the observer can be described by the equation

\[
\frac{d}{dt} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ \beta & -\beta \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ \beta \end{pmatrix} e
\]

Hence the covariance matrix, say \( P \), of the vector \((x \; \hat{x})^T\) can be computed from the Lyapunov equation

\[
0 = \begin{pmatrix} -\alpha & 0 \\ \beta & -\beta \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} -\alpha & \beta \\ 0 & -\beta \end{pmatrix} \\
+ \begin{pmatrix} \beta^2 r & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \beta^2 r \end{pmatrix}
\]

which leads to

\[
p_{11} = \frac{\beta^2 r}{2\alpha}, \quad p_{12} = \frac{\beta}{\alpha + \beta} \frac{\beta^2 r}{2\alpha}, \quad p_{22} = \frac{\alpha^2 + \alpha \beta + \beta^2 \beta r}{\alpha + \beta} \frac{2\alpha}{2\alpha}
\]

The error variance is

\[
E[x(t) - \hat{x}(t)]^2 = p_{11} - 2p_{12} + p_{22} = \frac{\beta(\alpha + 2\beta)}{2(\alpha + \beta)} r
\]

One can alternatively set up a state space model for \( x \) and \( \hat{x} \):

\[
\frac{d}{dt} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ -\alpha & -\beta \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} v + \begin{pmatrix} 0 \\ -\beta \end{pmatrix} e
\]

and solve the associated Lyapunov equation.

(d) Straightforward calculations give for \( \alpha = 4, \; \beta = 3 \):

\[
\begin{align*}
\text{Case (b)} & \quad P = r \\
\text{Case (c)} & \quad P = \frac{15}{7} r
\end{align*}
\]