Final exam: Control design

Date: December 14, 2001

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Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 1 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1, you will be accounted for the best performance of the homework assignments and Problem 1.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding materia: The textbooks ‘Reglerteori – flervariabla och olinjära metoder’, and ‘Reglerteknik – Grundläggande teori’, mathematical handbooks, calculators. Note that the following are not allowed: Exempelsamling med lösningar, copies of OH transparencies.

Good luck!
Problem 1

A DC servo, with transfer function \( G(s) = \frac{1}{s(s+1)} \), is represented in state space form as

\[
\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\
y = \begin{pmatrix} 1 & 0 \end{pmatrix} x
\]

An LQG based regulator, \( u = -L\dot{x} + mv \) is designed using the criterion

\[
V = \int_0^\infty (x^T(t)Q_1x(t) + u^T(t)Q_2u(t)) \, dt
\]

with

\[
Q_1 = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_2 = 1
\]

In the regulator \( \dot{x} \) is determined using a Kalman filter designed using some covariance matrices \( R_1 \) and \( R_2 \) as follows

<table>
<thead>
<tr>
<th>Case</th>
<th>( R_1 )</th>
<th>( R_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \begin{pmatrix} 0 &amp; 0 \ 0 &amp; 1 \end{pmatrix} )</td>
<td>0.01</td>
</tr>
<tr>
<td>2</td>
<td>( \begin{pmatrix} 0 &amp; 0 \ 0 &amp; 100 \end{pmatrix} )</td>
<td>0.01</td>
</tr>
<tr>
<td>3</td>
<td>( \begin{pmatrix} 0 &amp; 0 \ 0 &amp; 0.01 \end{pmatrix} )</td>
<td>0.01</td>
</tr>
</tbody>
</table>

In the diagrams below are displayed the sensitivity function \( S(i\omega) \) and the complementary sensitivity function \( T(i\omega) \) for these cases. The curves marked '0' refer to the case of an exact state feedback.

*Sensitivity functions*
Complementary sensitivity functions

\[ R_1 = \begin{pmatrix} 0 & 0 \\ 0 & 100 \end{pmatrix}, \quad R_2 = 1 \]

2 points

Problem 2

A multivariable system consists of two coupled parallel tanks, and has the dynamics

\[ \frac{d}{dt} x = \begin{pmatrix} -1 - \alpha & \alpha \\ \alpha & -1 - \alpha \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u \]

\[ y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x \]

The coefficient \( \alpha \) depends on the size of an opening between the two tanks, and \( \alpha > 0 \) holds.

(a) What is the transfer function of the system? 3 points

(b) Verify that one of the poles is always located in \( s = -1 \), no matter what value \( \alpha \) has. Where is the other pole located? 3 points

(c) Verify that the system has no zeros, no matter what value \( \alpha \) takes. 3 points

Problem 3

Consider the tank system in Problem 2. Set the parameter \( \alpha = 0.5 \). Assume the aim is to design a control system.
(a) Assume that decentralized control is to be used. With the numbering of the inputs and outputs in Problem 2, it may be possible to let $u_1$ be connected to $y_1$ and $u_2$ connected to $y_2$. A second alternative is to instead make the connections $u_1 - y_2$ and $u_2 - y_1$. Determine by using RGA for the static situation, which of the two alternatives that is preferable.  

3 points

(b) Determine a decoupling matrix $W_1(s)$ such that the system $\hat{G}(s) = G(s)W_1(s)$ is decoupled for $\omega = 0$.  

3 points

(c) Assume that the system is to be controlled by a diagonal proportional regulator  

$$U(s) = W_1(s) \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} (R(s) - Y(s))$$

where the regulator gains are set to $K_1 = K_2 = 10$. Determine the poles of the closed loop system, for two cases:

Case I: the regulator is applied without decoupling

Case II: the regulator is applied with the decoupling of part (b).  

5 points

Hint: For the specific regulator, part (c) is easiest solved by using simple calculations in the state space model.

Problem 4

Consider an oscillative system with transfer function  

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Design a feedback regulator using Internal Model Control. Will the regulator be integrating?  

5 points

Problem 5

Consider control of an integrating scalar system given by  

$$\dot{x} = u$$

$$y = x + w$$

Assume that the specifications for the control are given in terms of the weighting functions  

$$W_T = 0.5, \ W_S = \frac{a}{s}, \ W_u = b$$

a) Represent the extended system with artificial outputs $z$, see (10.4) in the textbook, in state space form.  

5 points

b) Are the conditions  

$$D^TM = 0, \ D^TD = I$$

satisfied? If not, describe what can be done about it?  

2 points
Problem 6

Consider the scalar system

\[
\begin{align*}
\dot{x} &= -x + u + v \\
y &= x + e
\end{align*}
\]

where the process noise \( v \) and the measurement noise \( e \) both have constant intensities \( \phi(\omega) \equiv 1 \).

a) Assume that the state \( x(t) \) is estimated using a standard observer

\[
\dot{\hat{x}} = -\hat{x} + u + K(y - \hat{x})
\]

with a constant gain \( K \). Determine the stationary variance, say \( V \), of the estimation error \( \hat{x} = x - \hat{x} \) as a function of \( K \).  

2 points

(b) Determine what value of the observer gain that minimizes \( V \). Let \( K^* \) denote this value of the gain. What is the minimum value of \( V \)?  

4 points

(c) What is the solution to the associated Riccati equation?  

2 points

(d) Assume next that the gain \( K^* \) is used, but that the observation process is improved by using a more accurate sensor, so that the measurement noise has intensity \( \Phi_e(\omega) \equiv 1/3 \). What is then the variance of the estimation error?  

2 points

(e) How much lower value of \( V \) can be obtained by re-optimizing the observer gain for the case treated in part (e)?  

2 points