Final exam: Computer-controlled system

Date: December 13, 2002

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Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 1 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1, you will be accounted for the best performance of the homework assignments and Problem 1.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as ‘Reglerteori – flervariabla och olinjära metoder’, ‘Reglerteknik – Grundläggande teori’, and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators. Note that the following are not allowed: Exempelsamling med lösningar, copies of OH transparencies.

Good luck!
Problem 1

A simplified model of an inverted pendulum on a cart is given as follows

\[
\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{41} & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{pmatrix} u
\]

\[y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x\]

Here, \(x_1\) denotes the angular displacement of the pendulum, \(x_2\) its angular velocity, \(x_3\) the position of the cart, \(x_4\) the velocity of the cart, and \(u\) is the applied force. The servo for controlling the cart is assumed fast in the model. Further, the matrix elements in the model above are

\[
a_{21} = \frac{3(m + M)g}{(m + 4M)L}, \quad a_{41} = -\frac{3mg}{m + 4M},
\]

\[
b_2 = -\frac{3}{L(m + 4M)} \quad b_4 = \frac{4}{m + 4M}
\]

where \(m\) is the mass of the pendulum, \(M\) is the mass of the cart, \(L\) is half the pendulum length, \(g\) is the constant of gravity.

a) Use the first two state equations to derive

\[(p^2 - a_{21})x_1 = b_2 u\]

2 points

b) Use also the remaining state equations to derive

\[p^2(p^2 - a_{21})x_3 = [b_4p^2 + (b_2a_{41} - b_4a_{21})]u\]

3 points

c) Determine the transfer functions from \(u\) to \(y_1\), and from \(u\) to \(y_2\). Also determine the poles and the zeros of these transfer functions. 4 points

Problem 2

Consider a system with no zero in the right half plane, and with transfer function \(G(s)\), which may be multivariable. Assume that a feedback regulator using the internal model principle is applied using \(\lambda\) tuning,

\[Q(s) = \frac{1}{(\lambda s + 1)^n}G^{-1}(s)\]

(a) Show that the loop gain \(F_y(s)G(s)\) contains an integrator. 4 points
(b) Assume that the process is controlled as
\[ u(t) = -F_y(s)[y(t) - F_r(s)r(t)] \]
with \( F_y(s) \) as above. Under what conditions on the feedforward link \( F_r(s) \) will the control be decoupled, so that each output component \( y_k(t) \) is effected only by one reference signal \( r_k(t) \) for \( k = 1, 2, \ldots \)? \hspace{1cm} \textbf{3 points}

\textbf{Problem 3}

Consider an unstable system with transfer function
\[ G(s) = \frac{1}{1 - sT}, \quad (T > 0) \]
Assume that it is to be controlled with an LQ regulator minimizing the criterion
\[ V = \int_0^\infty [y^2(t) + \rho u^2(t)] dt, \quad (\rho > 0) \]
(a) Determine the optimal regulator \( F_y(s) \). \hspace{1cm} \textbf{4 points}
(b) Determine the loop gain \( L_o(s) = G(s)F_y(s) \). \hspace{1cm} \textbf{2 points}
(c) Determine the cutoff frequency \( \omega_c \) defined by \( |L_o(\omega)| = 1 \). When \( \rho \) is varied, what is the smallest value \( \omega_c \) can take? \hspace{1cm} \textbf{4 points}

\textbf{Problem 4}

Consider a double tank with inflows to both the upper \( (u_1) \) and the lower \( (u_2) \) tank. Assuming both tank levels can be measured, a local model around a working point can be written as
\[ \dot{x} = \begin{pmatrix} -a & 0 \\ a & -a \end{pmatrix} x + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} u \]
\[ y = x \]
where \( a \) and \( b \) are positive constants.
(a) Determine the transfer function of the system, and its poles and zeros. \hspace{1cm} \textbf{2 points}
(b) Determine the singular values of the system. \hspace{1cm} \textbf{4 points}
Problem 5

Consider the feedback system below

\[ r = 0 \]

\[ \Sigma \]

\[ \begin{array}{c|c}
1 & -1 \\
\hline
\end{array} \]

\[ G(s) \]

\[ a \]

\[ b \]

\[ H(s) \]

where

\[ H(s) = s, \quad G(s) = \frac{1}{(s + 1)(s + 2)} \]

Use the circle criterion to design the feedback parameters \( a \) and \( b \) such that the closed loop system is guaranteed to be asymptotically stable.  

6 points

Problem 6

Consider the system

\[
\begin{align*}
\dot{x} &= u + v \\
y &= x + e
\end{align*}
\]

where \( v \) and \( e \) are scalar, independent and white noise processes, with intensities \( \beta^2r \) and \( r \), respectively. Consider LQG control of the system, using the criterion

\[ V = E[\alpha^2x^2(t) + u^2(t)] \]

The parameters \( \alpha \) and \( \beta \) are assumed to be positive.

(a) Determine the optimal regulator.  
4 points

(b) Determine the closed loop system in state space form, with \( v \) and \( e \) as inputs.  
3 points

(c) Determine, for the closed loop system, the variances \( Ex^2(t) \) and \( Eu^2(t) \).  
5 points