Final exam: Computer-Controlled System

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Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 1 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1, you will be accounted for the best performance of the homework assignments and Problem 1.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as ‘Reglerteori – flervari- abla och olinjära metoder’, ‘Reglerteknik – Grundläggande teori’, and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators. Note that the following are not allowed: Exempelsam- ling med lösningar, copies of OH transparencies.

Good luck!
Problem 1

Consider a system with three inputs and two outputs, having a transfer function

\[ G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix} \]

(a) Determine the poles and the zeros of the system. \hspace{1cm} 3 points

(b) The system can be represented in state space form as

\[ \dot{x} = Ax + Bu \\
\]

\[ y = Cx \]

with

\[ A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \hspace{0.5cm} C = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \]

and \( B \) a suitable matrix of dimension \( 3 \times 3 \). Determine the matrix \( B \).

6 points

Problem 2

The Volterra equations are popular for describing how prey and predators interplay.

(a) In a simple case, the model reads

\[ \dot{x} = x - xy \]

\[ \dot{y} = -y + xy \]

where \( x \) denotes the (amount of) prey, and \( y \) denotes the (amount of) predators. Determine the stationary points of this model, and there character. [It is OK to refer to the homework assignment for this part of the problem.] \hspace{1cm} 3 points

(b) Take it for given that the orbits for the model in part (a) are closed and represent period patterns. Let a typical orbit have period \( T \). What are the averaged values of \( x(t) \) and \( y(t) \) over one period, that is determine

\[ \bar{x} = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)dt, \hspace{0.5cm} \bar{y} = \frac{1}{T} \int_{t_0}^{t_0+T} y(t)dt \]

2 points

(c) In a somewhat more realistic case, the model of the interplay between predator and prey reads

\[ \dot{x} = x - xy - ex^2 \]

\[ \dot{y} = -y + xy \]

where the term \(-ex^2\) describes that the prey does not have an infinity supply of food. Even in absence of predators (that is, \( y = 0 \)), the number of prey will not grow without limits. For this model, determine the stationary solutions and their characters. Assume that \( 0 < e \leq 0.5 \). \hspace{1cm} 4 points
Problem 3

Sometimes in control design one assumes that the cutoff frequency $\omega_c$ is approximately equal to the bandwidth $\omega_B$. This problem is intended to illustrate the difference.

Consider a simple control system, with unit feedback as depicted below.

![Control System Diagram]

where $L(s)$ denotes the loop gain. For illustrating the answers, please detach the figure in the end of the exam, complement it with some circles, and hand it in along with your solution.

(a) The cutoff frequency $\omega_c$ is the solution to

$$|L(i\omega)| = 1$$

Show that the cutoff frequency appears at a point where the Nyquist curve cuts a certain circle. What is the center and the radius of the circle? Draw the circle on the attached figure. \textbf{2 points}

(b) Another characteristic frequency of the system is where the sensitivity function $S(s)$ has unit gain, that is $\omega_o$ being the solution to

$$|S(i\omega)| = 1$$

Show that the frequency $\omega_o$ occurs at a point where the Nyquist curve cuts a certain circle. What is the center and the radius of the circle? Draw the circle on the attached figure. \textbf{2 points}

(c) The bandwidth $\omega_B$ of the system is the frequency where the gain of the closed loop system transfer function $G_c(s)$ drops to $1/\sqrt{2}$ of the low frequency gain. Assume that the loop gain $L(s)$ is integrating. Then $G_c(0) = 1$, and the bandwidth is the solution to

$$|G_c(i\omega)| = \frac{1}{\sqrt{2}}$$

Show that the bandwidth occurs at a point where the Nyquist curve cuts a certain circle. What is the center and the radius of the circle? Draw the circle on the attached figure. \textbf{4 points}
Problem 4

Consider an undamped oscillator given by the state space model

\[
\begin{align*}
\dot{x} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x
\end{align*}
\]

It is to be controlled by state feedback of the form

\[
u = -\begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix} x + mr\]

where \( r \) is an external reference signal.

(a) How should \( m \) be designed, as a function of \( \ell_1 \) and \( \ell_2 \) if the output \( y \) should in stationarity follow the reference signal \( r \) without any static error? \hspace{1cm} 2 \text{ points}

(b) Assume that \( r \) is changed by a unit step, and that the maximal value of the input \( u \) occurs just as the step starts. Further, assume that the input must be bounded so that

\[|u(t)| \leq u_\omega, \text{ all } t\]

What does this condition imply for \( \ell_1 \) and \( \ell_2 \)? \hspace{1cm} 2 \text{ points}

(c) Assume that the feedback gains \( \ell_1 \) and \( \ell_2 \) are determined from an LQ problem with the weighting matrices

\[Q_1 = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 1\]

What does the condition treated in part (b) imply for the selection of the weight \( \rho \)? \hspace{1cm} 4 \text{ points}

Problem 5

Civ and Civerth were to apply internal model control for a system with transfer function

\[G(s) = \frac{1}{s + 1}\]

Civ chose

\[Q(s) = \frac{s + 1}{\lambda s + 1}\]

while Civerth’s choice was

\[Q(s) = \frac{s + 1}{(\lambda s + 1)^2}\]

Both of them chose the same value of the parameter \( \lambda \).

(a) Determine the regulator for the two cases. Will the regulators be integrating? \hspace{1cm} 3 \text{ points}
(b) What is the transfer function from the reference signal $r$ to the output $y$ in the two cases? \hspace{1cm} 2 \text{ points} \\

\textbf{Problem 6} \\

An engineer was given the task to design a Kalman filter for tracking a process variable from noisy measurements.

(a) First the engineer tried the simple model (called $M_1$ in what follows) 

\[
\begin{align*}
\dot{x} & = v \\
y & = x + e 
\end{align*}
\]

where $v$ and $e$ are scalar and independent white noise processes, with intensities $\beta^2 r$ and $r$, respectively. Determine the Kalman filter based on this model. The filter so designed will be denoted $F_1$. How large is the estimation error variance 

\[
E[x(t) - \hat{x}(t)]^2 
\]

assuming that the model $M_1$ describes the data? \hspace{1cm} 2 \text{ points} \\

(b) Then the engineer got more information, and realized that a more realistic model of the signal to be estimated is the following one (denoted $M_2$ in what follows) 

\[
\begin{align*}
\dot{x} & = -\alpha x + v \quad (\alpha > 0) \\
y & = x + e 
\end{align*}
\]

where $v$ and $e$ are noise processes as before. Determine the Kalman filter (called $F_2$) that is based on the model $M_2$. What is the error variance in this case? \hspace{1cm} 2 \text{ points} \\

(c) Before the engineer implemented the filter $F_2$, he tried the filter $F_1$. Determine the error variance when the filter $F_1$ is used, and the data are described by the model $M_2$. \hspace{1cm} 6 \text{ points} \\

(d) Evaluate the error variances in the cases (b) and (c) numerically when $\alpha = 4$ and $\beta = 3$. \hspace{1cm} 1 \text{ point}
Figure to be used for the solution of Problem 3

Figure 1: Nyquist curve for the loop gain $L(i\omega)$. 