Final exam: Computer-controlled systems

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Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 3 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 3, you will be accounted for the best performance of the homework assignments and Problem 3.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding materiaal: Textbooks in automatic control (such as ‘Reglerteori – flervari- abla och olinjära metoder’, ‘Reglerteknik – Grundläggande teori’, and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators. Note that the following are not allowed: Exempelsam- ling med lösningar, copies of OH transparencies.

Good luck!
Problem 1

Consider a two-input, two-output system with the transfer function

\[
G(s) = \begin{pmatrix}
\frac{1}{s+2} & \frac{10}{s+1} \\
\frac{1}{s+5} & \frac{5}{s+3}
\end{pmatrix}
\]

(a) Determine RGA(\(G(0)\)). 
4 points

(b) Which input-output pairing should be preferred? 
2 points

Problem 2

Assuming no disturbances are present, the following model is assumed to describe a dynamic system.

\[
\frac{d}{dt}y(t) + y(t) = u(t).
\]

The system should be controlled by a \(H_2\) designed regulator, where the following weights apply:

\[
W_s(s) = \frac{1}{s}, \quad W_T(s) = 1 \quad \text{och} \quad W_u = 1.
\]

(a) Determine a state space model for the extended system in the standard form

\[
\frac{d}{dt}x(t) = Ax(t) + Bu(t) + Nw(t) \\
z(t) = Mx(t) + Du(t) \\
y(t) = Cx(t) + w(t)
\]

4 points

(b) Determine the transfer function \(F_y(p)\), where the resulting regulator is \(u(t) = -F_y(p)y(t)\) 
4 points

Problem 3

(a) Consider a multivariable system with the transfer function

\[
G(s) = \begin{pmatrix}
\frac{1}{s+1} & \frac{2}{s+2} \\
\frac{3}{s+1} & \frac{-2}{s+1}
\end{pmatrix}
\]

Show that the system can be represented in state space form as

\[
\dot{x} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} x + \begin{pmatrix}
1 & 0 \\
0 & 2 \\
3 & 0 \\
0 & -2
\end{pmatrix} u \\
y = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix} x
\]

3 points
(b) What is the pole polynomial of the transfer function $G(s)$ given in part (a)?

2 points

(c) Consider the state space representation, say $S_1$, in part (a). Show how it can be reduced to another state space representation, say $S_2$. The new representation $S_2$ should have smaller order than $S_1$, and be both controllable and observable.

4 points

Problem 4

Consider a rocket in space, which we model as a double integrator for the movement in one direction. The acceleration is due to some process noise, so with the state variables $x_1 = y$, $x_2 = \dot{y}$ the state space model is

\[
\begin{align*}
\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + e
\end{align*}
\]

We assume for the time being that the position $y$ is measured with some error $e$.

Assume that the spectra of the noise sources are

\[
\phi_v = R_1 = \gamma^4, \quad \phi_e = R_2 = 1
\]

(a) Determine the Kalman filter. Determine also the covariance matrix of the estimation error

\[P = E\hat{x}(t)\hat{x}^T(t)\]

4 points

(b) Assume next that a second sensor is added, so that also the velocity $\dot{y}$ is measured. The output equation is then changed to

\[y = x + e\]

where now $e$ is a two-dimensional vector with spectrum

\[
\phi_e = R_2 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix}
\]

Determine the optimal Kalman filter in this case. Show that in this case the covariance matrix of the estimation error is

\[P = E\hat{x}(t)\hat{x}^T(t) = \frac{1}{2} \begin{pmatrix} \sqrt{3}\gamma & \gamma^2 \\ \gamma^2 & \sqrt{3}\gamma^3 \end{pmatrix}\]

3 points
Problem 5

(a) Consider the system

\[ G(s) = \frac{K(1-s)}{(s+1)^2} \]

What is the utmost highest possible cut-off frequency that can be chosen when controlling this system with the feedback \( u(t) = -y(t) \)? 3 points

(b) How high can the gain \( K \) be chosen, if the system is controlled with unit feedback \( (u = -y) \) before losing stability? 2 points

(c) Assume that it is imposed in the design that there must be a phase margin of at least 45° degrees. How does that effect the questions treated in parts (a) and (b)? 4 points

Problem 6

Consider the feedback system in the figure below.

\[ e \xrightarrow{f(e)} u \xrightarrow{G(s)} y \]

\[ -1 \]

Introduce the state variables \( x_1 = y, \; x_2 = \dot{y} \).

(a) Assume that \( G(s) = 1/s^2 \) and that the nonlinearity is an ideal relay, so

\[ u(e) = \begin{cases} 
1 & \text{if } e > 0 \\
0 & \text{if } e = 0 \\
-1 & \text{if } e < 0 
\end{cases} \]

Sketch the phase portrait. Will the trajectories be closed orbits? 4 points

(b) Assume that \( G(s) = 1/s^2 \) and that the relay has a deadzone, so

\[ u(e) = \begin{cases} 
1 & \text{if } e > a \\
0 & \text{if } -a \leq e \leq a \\
-1 & \text{if } e < -a 
\end{cases} \]

Sketch the phase portrait. Will the trajectories be closed orbits? 3 points
(c) Assume that \( G(s) = 1/s^2 \) and that the relay has a hysteresis, so

![Diagram showing hysteresis characteristic]

Sketch the phase portrait. Will the trajectories be closed orbits? 4 points