Final exam: Computer-controlled system (Datorbaserad styrning, 1TV492, 1TS250)

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Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 5 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 5 you will be accounted for the best performance of the homework assignments and Problem 5.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding materia: Textbooks in automatic control (such as ‘Reglerteori – flervari- 
abla och olinjära metoder’, ‘Reglerteknik – Grundläggande teori’, and others),
mathematical handbooks, collection of formulas (formelsamlingar), textbooks in 
mathematics, calculators. Note that the following are not allowed: Exempelsam-
ling med lösningar, copies of OH transparencies.

Good luck!
Problem 1

Consider the system

\[
\dot{x} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -2
\end{pmatrix} x + \begin{pmatrix}
1 & 1 \\
1 & 1 \\
3 & 0
\end{pmatrix} u
\]

\[
y = \begin{pmatrix}
1 & 2 & 1 \\
2 & 1 & 0
\end{pmatrix} x
\]

(a) Is the system controllable? 2 points

(b) Is the system observable? 2 points

(c) Determine the poles of the system. 2 points

Problem 2

Use internal model control with \( \lambda \)-tuning to design a regulator for a DC motor with the transfer function

\[ G(s) = \frac{1}{s(s + 1)} \]

(a) Express the regulator using \( \lambda \) as a parameter. Is the regulator integrating? 2 points

(b) Determine the loop gain? 2 points

(c) Choose \( \lambda \) so that the controlled system has cut-off frequency \( \omega_c \). 2 points

(d) Determine the phase margin! 2 points

Problem 3

Let \( x(t) \) be a stationary continuous-time process with covariance function \( r_x(\tau) \) and spectral density \( \phi_x(\omega) \).

(a) Consider the process

\[ y(t) = \frac{x(t + h) - x(t - h)}{2h} \]

which is an approximate time-derivative of \( x(t) \).

Show that the covariance function \( r_y(\tau) \) of \( y(t) \) is

\[
r_y(\tau) = \frac{2r_x(\tau) - r_x(\tau + 2h) - r_x(\tau - 2h)}{4h^2}
\]

3 points
(b) Show that the spectrum of $y(t)$ is

$$\phi_y(\omega) = \phi_x(\omega) \frac{\sin^2(\omega h)}{h^2}$$

3 points

(c) What is the spectrum of the true derivative $\dot{x}(t)$?

1 point

Problem 4

Consider an unstable system with transfer function

$$G(s) = \frac{1}{1 - sT}, \quad (T > 0)$$

Assume that it is to be controlled with an LQ regulator minimizing the criterion

$$V = \int_0^\infty [y^2(t) + \rho u^2(t)] dt, \quad (\rho > 0)$$

(a) Determine the optimal regulator $F_y(s)$.

4 points

(b) Determine the loop gain $L_o(s) = G(s)F_y(s)$.

2 points

(c) Determine the cutoff frequency $\omega_c$ defined by $|L_o(i\omega)| = 1$. When $\rho$ is varied, what is the smallest value $\omega_c$ can take?

4 points

Problem 5

(a) Consider a harmonic oscillator with transfer function

$$G(s) = \frac{\omega_o^2}{s^2 + \omega_o^2}$$

Represent it in state space form as

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \omega_o^2 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Determine the state space form of the sampled version of this system, when the sampling interval equals $h$.

3 points

(b) Determine the pulse transfer function ("överföringsoperatorn") of the sampled system of part (a).

3 points

(c) Given a discrete-time system with the pulse transfer function

$$H(q) = \frac{1}{q + 1}$$

Can this system occur by sampling a second order system with the sampling interval being $h$? Motivate your answer!

3 points
Problem 6

Consider the control system below, where a DC motor is controlled by using a saturizing amplifier.

\[
\begin{align*}
G(s) &= \frac{K}{s(s + 1)} \quad (K > 0) \\

f(e) &= \begin{cases} 
-1 & e < -1 \\
\quad e & -1 < e < 1 \\
\quad 1 & 1 < e 
\end{cases}
\end{align*}
\]

(a) Use the circle criterion to find sufficient conditions on $K$ for the closed loop system to be stable. 4 points

(b) Write the closed loop system on state space form using $y$ and $\dot{y}$ as state variables. 2 points.

(c) Analyse stability of the closed loop system. Use the state space model derived in part (b). Try a Lyapunov function of the form

\[V(x) = \frac{1}{2} x_2^2 + Kg(x_1)\]

where $g(x_1)$ is some suitable function. 4 points
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— Answers and brief solutions

Problem 1

(a) The controllability matrix becomes

\[
W_c = \begin{pmatrix}
1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 \\
3 & 0 & -6 & 0 & 12 & 0
\end{pmatrix}
\]

The first two rows are identical and therefore linearly dependent. The rank of \( W_c \) is 2. The system is not controllable.

(b) The beginning of the observability matrix is

\[
W_o = \begin{pmatrix}
1 & 2 & 1 \\
2 & 1 & 0 \\
-1 & -2 & -2
\end{pmatrix}
\]

As the upper three rows are linearly independent, it is not necessary to compute further rows to conclude that \( W_o \) has rank 3. The system is observable.

(c) The \( A \) matrix has two eigenvalues in \( s = -1 \) and one in \( s = -2 \). As the system is not controllable, it is not possible to say that this is precisely the poles. Computing the transfer function gives

\[
G(s) = C(sI - A)^{-1} B
\]

\[
= \ldots = \frac{1}{s + 1} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{s + 2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{3(2s+3)}{(s+1)(s+2)} \\
\frac{3}{s+1}
\end{pmatrix}
\]

The minors of order 1 are: \( \frac{3(2s+3)}{(s+1)(s+2)} \) and \( \frac{3}{s+1} \).

There is one minor of order 2: \( \det G(s) = \frac{3}{(s+1)(s+2)} \).

The pole polynomial is hence \( (s + 1)(s + 2) \). The system has one pole in \( s = -1 \) and one pole in \( s = -2 \).
Problem 2

(a)

\[ Q(s) = \frac{1}{(1 + \lambda s)^2 s(s + 1)} \]

\[ F_y(s) = \frac{Q(s)}{1 - Q(s)G(s)} = \frac{s(s + 1)}{(1 + \lambda s)^2} \frac{1}{1 - \frac{s(s + 1)}{(1 + \lambda s)^2} \frac{1}{s(s + 1)}} = \frac{s(s + 1)}{(1 + \lambda s)^2 - 1} = \frac{s + 1}{\lambda^2 s^2 + 2\lambda s} = \frac{s + 1}{\lambda s^2 + 2\lambda} \]

As \( F_y(0) \neq \infty \), the regulator is not integrating.

(b) The loop gain becomes

\[ L(s) = F_y(s)G(s) = \frac{s + 1}{\lambda^2 s + 2\lambda} \frac{1}{s(s + 1)} = \frac{1}{\lambda^2 s^2 + 2\lambda s} \]

(c) The cut-off frequency \( \omega_c \) is defined by \(|L(i\omega_c)| = 1\). Hence,

\[ |-\omega_c^2 + 2i\omega_c| = 1 \Rightarrow \alpha^4 + 4\alpha^2 - 1 = 0 \Rightarrow \alpha^2 = -2 + \sqrt{5} \Rightarrow \alpha = \sqrt{\sqrt{5} - 2} \approx 0.486. \]

Hence we have

\[ \lambda = \frac{0.486}{\omega_c} \]

(d) The phase margin is

\[ \varphi_m = 180^\circ + \arg L(i\omega_c) = 180^\circ + \frac{1}{-\alpha^2 + 2i\alpha} = 180^\circ - \arg(-\alpha^2 + 2i\alpha) = \arg(-\frac{\alpha^2}{2}) = \frac{\alpha}{2} = \arg\left(\frac{2}{\sqrt{\sqrt{5} - 2}}\right) = \arctan(2\sqrt{\sqrt{5} + 2}) \approx 76.34^\circ \]

Problem 3

(a) The covariance function is easily found:

\[ r_y(\tau) = E_y(t + \tau)y(t) = \frac{1}{4h^2} E[x(t + \tau + h) - x(t + \tau - h)][x(t + h) - x(t - h)] \]

\[ = \frac{1}{4h^2}[r_x(\tau) - r_x(\tau + 2h) - r_x(\tau - 2h) + r_x(\tau)] \]

\[ = \frac{1}{4h^2}[2r_x(\tau) - r_x(\tau + 2h) - r_x(\tau - 2h)] \]
(b) The spectrum becomes

\[
\phi_y(\omega) = \int_{-\infty}^{\infty} r_y(\tau) e^{-i\omega \tau} d\tau \\
= \frac{1}{4h^2} \int_{-\infty}^{\infty} [2r_x(\tau) - r_x(\tau + 2h) - r_x(\tau - 2h)] e^{-i\omega \tau} d\tau \\
= \frac{1}{4h^2} \left[ 2\phi_x(\omega) - \int_{-\infty}^{\infty} r_x(\tau') e^{-i\omega \tau' + i\omega 2h} d\tau' - \int_{-\infty}^{\infty} r_x(\tau') e^{-i\omega \tau' - i\omega 2h} d\tau' \right] \\
= \frac{1}{4h^2} \phi_x(\omega) \left[ 2 - e^{i2\omega h} - e^{-i2\omega h} \right] \\
= \frac{1}{4h^2} \phi_x(\omega) [2 - 2\cos(2\omega h)] \\
= \frac{1}{4h^2} \phi_x(\omega) \left[ 2 - 2 \left\{ 1 - 2 \sin^2(\omega h) \right\} \right] \\
= \phi_x(\omega) \frac{\sin^2(\omega h)}{h^2}
\]

(c)

\[
\phi_z(\omega) = |i\omega|^2 \phi_x(\omega) = \omega^2 \phi_x(\omega)
\]

Note that \(\phi_y(\omega) \approx \phi_z(\omega)\) for small \(h\) (or small \(\omega\)). Further, \(\phi_y(\omega)\) and \(\phi_z(\omega)\) will differ significantly for large \(\omega\).

**Problem 4**

(a) 

\[
\dot{x} = \frac{1}{T} x - \frac{1}{T} u \\
y = x
\]

Riccati equation

\[
0 = 2 \frac{1}{T} S + 1 - \frac{S^2}{T^2 \rho} \quad L = -\frac{S}{T \rho}
\]

\[
S^2 - 2ST \rho - T^2 \rho = 0
\]

with solution

\[
S = T \rho \pm \sqrt{T^2 \rho^2 + T^2 \rho} \\
= T \rho [1 + \sqrt{1 + 1/\rho}] \\
F_y(s) = L = -1 - \sqrt{1 + 1/\rho}
\]

(b)

\[
L_\alpha(s) = \frac{L}{1 - sI}
\]
(c) 
\[ |L_0(i\omega_c)| = 1 \Rightarrow \]
\[ 1 + \omega_c^2 T^2 = |L|^2 = 1 + (1 + 1/\rho) + 2\sqrt{1 + 1/\rho} \]
\[ \omega_c^2 = \frac{1}{T^2}[1 + 1/\rho + 2\sqrt{1 + 1/\rho}] \geq \frac{3}{T^2} \text{ with equality for } \rho \to \infty \]

**Problem 5**

(a) Compute first the matrix exponential.
\[ e^{At} = \mathcal{L}^{-1} \left( \begin{array}{cc} s & -1 \\ \omega_c^2 & s \end{array} \right)^{-1} = \mathcal{L}^{-1} \frac{1}{s^2 + \omega_c^2} \left( \begin{array}{cc} s & 1 \\ -\omega_c^2 & s \end{array} \right)^{-1} \]
\[ = \left( \begin{array}{cc} \cos(\omega_c t) & \frac{1}{\omega_c} \sin(\omega_c t) \\ -\omega_c \sin(\omega_c t) & \cos(\omega_c t) \end{array} \right) \]

Set
\[ C = \cos(\omega_c h), \quad S = \sin(\omega_c h) \]

Then
\[ F = \left( \begin{array}{cc} C & \frac{1}{\omega_c} S \\ -\omega_c S & C \end{array} \right) \]
\[ G = \int_0^h e^{At} B ds = \int_0^h \left( \begin{array}{cc} \omega_c \sin(\omega_c s) \\ \omega_c^2 \cos(\omega_c s) \end{array} \right) ds = \left( \begin{array}{cc} 1 - C \\ \omega_c S \end{array} \right) \]

(b) 
\[ H(q) = \left( \begin{array}{cc} 1 & 0 \\ \omega_c S & q - C \end{array} \right)^{-1} \left( \begin{array}{cc} 1 - C \\ \omega_c S \end{array} \right) \]
\[ = \frac{1}{(q - C)^2 + S^2} \left( \begin{array}{cc} q - C & \frac{1}{\omega_c} S \\ \omega_c S & 1 - C \end{array} \right) \]
\[ = \frac{(1 - C)(q + 1)}{q^2 - 2qC + 1} \]

(c) Use the results of parts (a) and (b). Consider a continuous-time system with the transfer function
\[ G(s) = \frac{K\omega_c^2}{s^2 + \omega_c^2} \]
(where the parameters $K$ and $\omega_c$ are to be determined). The pulse transfer function of this system is, according to part (b)
\[ H(q) = \frac{K(1 - C)(q + 1)}{q^2 - 2qC + 1} \]

Now choose in particular
\[ K = 0.5, \quad \omega_c = \pi/h \]

Then, $C = -1$ and
\[ H(q) = \frac{0.5 \times 2(q + 1)}{q^2 + 2q + 1} = \frac{1}{q + 1} \]
Problem 6

(a) The nonlinearity gives

$$k_1 \leq \frac{|f(e)|}{e} \leq k_2$$

leading to $k_1 = 0, \ k_2 = 1$. Hence the circle in the circle criterion will be the area to the left of the line $\Re(s) = -1$. The (sufficient) stability condition is therefore that the Nyquist curve lies to the right of this line, that is

$$\Re(G(i\omega)) \geq -1, \ \forall \omega \Rightarrow \Re\left(\frac{K(-i\omega)(-i\omega + 1)}{\omega^2(\omega^2 + 1)}\right) \geq -1, \ \forall \omega
$$

$$\Rightarrow \left(\frac{-K\omega^2}{\omega^2(\omega^2 + 1)}\right) \geq -1, \ \forall \omega
$$

$$\Rightarrow \frac{K}{\omega^2 + 1} \leq 1, \ \forall \omega \Rightarrow K < 1$$

(b) Set $x_1 = y, \ x_2 = \dot{y}$. Then

$$\dot{x}_1 = x_2, \ \dot{x}_2 = \ddot{y} = -x_2 - Kf(x_1)$$

(c) Try a Lyapunov function of the form

$$V(x) = \frac{1}{2}x_2^2 + Kg(x_1)$$

Then one gets

$$\dot{V} = x_2\dot{x}_2 + K\frac{\partial g}{\partial x_1}\dot{x}_1$$

$$= x_2[-x_2 - Kf(x_1)] + K\frac{\partial g}{\partial x_1}x_2$$

$$= -x_2^2 + Kx_2\left[-f(x_1) + \frac{\partial g}{\partial x_1}\right]$$

Now choose $g(e)$ so that

$$\frac{\partial g}{\partial e} = f(e)$$

Then we have $\dot{V} = -x_2^2 \leq 0$. Further, there is no solution (except $x \equiv 0$) that satisfies $\dot{V} = 0$. Hence the system is stable, and all solutions converge to $x = 0$. The precise choice of the function $g(e)$ is a primitive function of $f(e)$:

$$g(e) = \begin{cases} 
0.5 + (-e - 1) & e < -1 \\
0.5e^2 & -1 < e < 1 \\
0.5 + (e - 1) & 1 < e
\end{cases}$$