Yule-Walker equations

Consider the case of an AR process

$$y(t) + a_1 y(t-1) + \cdots + a_n y(t-n) = e(t), \qquad \mathbf{E}e^2(t) = \lambda$$

Note that $y(t)$ can be viewed as a linear combination of all old values of the noise. By multiplying $y(t)$ by a delayed value of the process, say $y(t-\tau)$, $\tau \geq 0$, and applying the expectation operator, one obtains

$$\mathbf{E}y(t-\tau)[y(t) + a_1 y(t-1) + \cdots + a_n y(t-n)] = \mathbf{E}y(t-\tau)e(t)$$

that can be rewritten as

$$R_y(\tau) + a_1 R_y(\tau-1) + \cdots + a_n R_y(\tau-n) = \begin{cases} 0, & \tau > 0 \\ \lambda, & \tau = 0 \end{cases}$$

Using this information, one can construct the following system of equations for determining the convariance functions $R_y(0)$, $R_y(1)$, $\ldots$, $R_y(n)$:

$$\underbrace{\begin{pmatrix} 1 & a_1 & \ldots & a_n \\ a_1 & 1+a_2 & a_n & 0 \\ \vdots & & \ddots & \vdots \\ a_n & a_{n-1} & \ldots & 1 \end{pmatrix}}_{A} \begin{pmatrix} R_y(0) \\ \vdots \\ \\ R_y(n) \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1}$$

$$A = \begin{pmatrix} 1 & & \ldots & 0 \\ a_1 & 1 & & \\ \vdots & & \ddots & \\ a_n & a_{n-1} & & 1 \end{pmatrix} + \begin{pmatrix} 0 & a_1 & \ldots & a_n \\ & \vdots & a_n & 0 \\ \vdots & a_n & \ddots & \\ 0 & 0 & & 0 \end{pmatrix}$$

Once $R_y(0)$, $R_y(1)$, $\ldots$, $R_y(n)$ are known, (1) can be iterated (for $\tau = n+1$, $n+2$, $\ldots$) to find further covariance elements.