(1) a) Lägg till en etta i φ-vektorn och skatta konstanten.

$$\varphi(t) = \begin{bmatrix} -y(t-1) \\ \vdots \\ u(t-1) \\ \vdots \\ 1 \end{bmatrix} \qquad \theta = \begin{bmatrix} a_1 \\ \vdots \\ b_1 \\ \vdots \\ K \end{bmatrix}$$

b) Endast ARX och AR.

OE kräver iterativa lösare.

c) Tag en ARX-process med

$$A(q) = 1 - 2q^{-1}, \quad B(q) = 1$$

$$y(t) = \frac{1}{1-2q^{-1}} u(t) + \frac{1}{q-2q^{-1}} e(t) = \frac{q}{q-2} u(t) + \frac{q}{q-2} e(t)$$

har pol. i $z=2 \Rightarrow$ instabilt system.

Optimala prediktorn ges av

$$\hat{y}(t) = (1 - A(q))y(t) + B(q)u(t) =$$

$$= 2q^{-1}y(t) + u(t) = \frac{2}{q} y(t) + u(t)$$

Har pol i $z=0 \Rightarrow$ stabil prediktor!

För OE gäller att prediktorn ges

av modellen $\hat{y}(t) = \frac{B(q)}{F(q)} u(t)$. Om denna

är stabil är naturligtvis även $y(t) = \frac{B(q)}{F(q)} u(t) + e(t)$

stabil!

(2)

Från definitionen (c.18) har vi att

$$\phi(\omega) = T \sum_{k=-\infty}^{\infty} R(kT) e^{-i\omega kT}.$$

Med $T=1$ får vi

$$\phi(\omega) = \sum_{k=-\infty}^{\infty} R(k) e^{-i\omega k}.$$

Vi har

$$\phi(\omega) = 1 + \cos(\omega) = 1 + \frac{e^{i\omega} + e^{-i\omega}}{2} = \frac{1}{2} e^{i\omega} + 1 + \frac{1}{2} e^{-i\omega}$$

$$\therefore \begin{cases} R(0) = 1 \\ R(1) = R(-1) = \frac{1}{2} \\ R(k) = 0 \quad |k| \geq 2 \end{cases}$$

(3)

a) $\Phi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\Phi^T\Phi = [1 \quad 1]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2$, $Y = \begin{bmatrix} y(1) \\ y(2) \end{bmatrix}$, $R = \begin{bmatrix} \lambda(1) & 0 \\ 0 & \lambda(2) \end{bmatrix}$

$$\underline{\underline{\hat{c}}} = \hat{\theta} = [\Phi^T\Phi]^{-1}\Phi^T Y = [2]^{-1}[1 \quad 1]\begin{bmatrix} y(1) \\ y(2) \end{bmatrix} = \underline{\underline{\frac{1}{2}(y(1)+y(2))}}$$

Variansen ges av

se kompendium

$$\underline{\underline{\operatorname{Var}\hat{c}}} = \operatorname{cov}\hat{\theta} \overset{(13)}{=} (\Phi^T\Phi)^{-1}\Phi^T R \Phi (\Phi^T\Phi)^{-1} =$$

$$= (2)^{-1}[1 \quad 1]\begin{bmatrix} \lambda(1) & 0 \\ 0 & \lambda(2) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(2)^{-1} =$$

$$= \frac{1}{4}(\lambda(1) + \lambda(2)) = \frac{1}{4}(1+100) = \underline{\underline{\frac{101}{4}}}$$

b) Optimala skattningen då R är känd är BLUE-skattningen.

$$\underline{\underline{\hat{\theta}_{BLUE}}} = (\Phi^T R^{-1}\Phi)^{-1}\Phi^T R^{-1} Y = \left[[1 \quad 1]\begin{bmatrix} \lambda(1) & 0 \\ 0 & \lambda(2) \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]^{-1}[1 \quad 1]\begin{bmatrix} \lambda(1) & 0 \\ 0 & \lambda(2) \end{bmatrix}^{-1}\begin{bmatrix} y(1) \\ y(2) \end{bmatrix}$$

$$= \left[\frac{1}{\lambda(1)} + \frac{1}{\lambda(2)}\right]^{-1}\left(\left[\frac{1}{\lambda(1)} \quad \frac{1}{\lambda(2)}\right]\right)\begin{bmatrix} y(1) \\ y(2) \end{bmatrix} =$$

$$= \left(\frac{y(1)}{\lambda(1)} + \frac{y(2)}{\lambda(2)}\right) \Big/ \left(\frac{1}{\lambda(1)} + \frac{1}{\lambda(2)}\right) = \frac{\left(\frac{y(1)}{1} + \frac{y(2)}{100}\right)}{\frac{1}{1} + \frac{1}{100}} = \underline{\underline{\frac{100}{101}\left(y(1) + \frac{y(2)}{100}\right)}}$$

$\text{cov}\,\hat{\theta}_{BLUE} = \left(\hat{\Phi}^T R^{-1} \hat{\Phi}\right)^{-1} = \text{se ovan} =$

$$= \frac{1}{\frac{1}{\lambda(1)} + \frac{1}{\lambda(2)}} = \frac{100}{101}$$

(4) För problemet gäller att

$$\varphi(t) = \begin{bmatrix} u(t) \\ u(t-1) \end{bmatrix}, \quad \theta = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

och skattningen ges av

$$\hat{\theta} = \left[\sum_{t=2}^{4} \varphi(t)\varphi^T(t)\right]^{-1} \sum_{t=2}^{4} \varphi(t) y(t).$$

Kovariansen i skattningen ges av

$$\operatorname{cov} \hat{\theta} = \lambda \left[\sum_{t=2}^{4} \varphi(t)\varphi^T(t)\right]^{-1}$$

Vi är alltså intresserade av att studera

$$\sum_{t=2}^{4} \varphi(t)\varphi^T(t) = \begin{bmatrix} \sum_{t=2}^{4} u^2(t) & \sum_{t=2}^{4} u(t)u(t-1) \\ \sum_{t=2}^{4} u(t)u(t-1) & \sum_{t=2}^{4} u^2(t-1) \end{bmatrix} \triangleq R$$

1.

$$\sum_{t=2}^{4} u^2(t) = (-10)^2 + (10)^2 + (-10)^2 = 300$$

$$\sum_{t=2}^{4} u^2(t-1) = (10)^2 + (-10)^2 + (10)^2 = 300$$

$$\sum_{t=2}^{4} u(t)u(t-1) = (-10)\cdot 10 + 10\cdot(-10) + (-10)\cdot 10 = -300$$

$R = 300 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ singulär! θ kan ej skattas!

2.

$$\sum_{t=2}^{4} u^2(t) = 2^2 + 0^2 + 0^2 = 4$$

$$\sum_{t=2}^{4} u^2(t-1) = 0^2 + 2^2 + 0^2 = 4$$

$$\sum_{t=2}^{4} u(t)u(t-1) = 0 \cdot 2 + 2 \cdot 0 + 0 \cdot 0 = 0$$

$$R = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\underline{\underline{\text{cov}\,\hat{\theta}}} = \lambda \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}^{-1} = \underline{\underline{\frac{\lambda}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}}$$

3.

$$\sum_{t=2}^{4} u^2(t) = 0^2 + 0^2 + 0^2 = 0$$

$$\sum_{t=2}^{4} u^2(t-1) = 100^2 + 0^2 + 0^2 = 10000$$

$$\sum_{t=2}^{4} u(t)u(t-1) = 0 \cdot 100 + 0 \cdot 0 + 0 \cdot 0 = 0$$

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 10000 \end{bmatrix}$$ singulär!

<u>θ kan ej skattas!</u>

4.

$$\sum_{t=2}^{4} u^2(t) = 1^2 + (-1)^2 + (-1)^2 = 3$$

$$\sum_{t=2}^{4} u^2(t-1) = 1^2 + 1^2 + (-1)^2 = 3$$

$$\sum_{t=2}^{4} u(t)u(t-1) = 1\cdot 1 + (-1)\cdot 1 + (-1)\cdot(-1) = 1$$

$$R = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\operatorname{cov} \hat{\theta} = \lambda R^{-1} = \lambda \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{\lambda}{9-1} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} =$$

$$= \frac{\lambda}{8} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

⑤

a) $\hat{y}(t) = b_1 u(t-1) + b_2 u(t-2) = \varphi^T(t)\theta$

$$\varphi(t) = \begin{bmatrix} u(t-1) \\ u(t-2) \end{bmatrix}, \quad \theta = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$$\operatorname{cov}(\hat{\theta}) \to \frac{\lambda}{N}\left[E\,\varphi(t)\varphi^T(t)\right]^{-1} = \frac{\lambda}{N}\begin{bmatrix} \overset{=\delta_u}{R_u(0)} & R_u(1) \\ \underset{=0}{R_u(1)} & R_u(0) \end{bmatrix}^{-1} =$$

$$= \frac{\lambda}{N}\begin{bmatrix} \delta_u & 0 \\ 0 & \delta_u \end{bmatrix}^{-1} = \frac{\lambda}{N}\begin{bmatrix} 1/\delta_u & 0 \\ 0 & 1/\delta_u \end{bmatrix}$$

$$\therefore \operatorname{var}(\hat{b}_1) = \operatorname{var}(\hat{b}_2) = \frac{\lambda}{N\delta_u}$$

b) $\hat{y}(t) = b(u(t-1) + u(t-2)) = \varphi^T(t)\theta$

$$\varphi(t) = u(t-1) + u(t-2), \quad \theta = b.$$

$$\underline{\underline{\operatorname{var}(\hat{b})}} = \operatorname{cov}\hat{\theta} \to \frac{\lambda}{N}\left[E\,\varphi(t)\varphi^T(t)\right]^{-1} =$$

$$= \frac{\lambda}{N}\left(E\,(u(t-1) + u(t-2))^2\right)^{-1} =$$

$$= \frac{\lambda}{N}\left[E(u^2(t-1) + 2u(t-1)u(t-2) + u^2(t-2))\right]^{-1} =$$

$$= \frac{\lambda}{N}\left(\delta_u + 0 + \delta_u\right)^{-1} = \underline{\underline{\frac{\lambda}{2N\delta_u}}}$$

(6)

$\underline{n=2}$: $\varphi(t) = \begin{bmatrix} u(t-1) \\ u(t-2) \end{bmatrix}$, $\theta = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\hat{\theta} \to [\underbrace{E\varphi(t)\varphi^T(t)}_{R}]^{-1} E\varphi(t)y(t)$$

$$R = E\varphi(t)\varphi^T(t) = \begin{bmatrix} R_u(0) & R_u(1) \\ R_u(1) & R_u(0) \end{bmatrix} = \frac{A^2}{2}\begin{bmatrix} 1 & \cos(\omega) \\ \cos(\omega) & 1 \end{bmatrix}$$

R har full rang, ty $\cos(\omega) \neq 1 \Rightarrow$

systemet är identifierbart.

$\underline{n=3}$ $\varphi(t) = \begin{bmatrix} u(t-1) \\ u(t-2) \\ u(t-3) \end{bmatrix}$

$$R = \begin{bmatrix} R_u(0) & R_u(1) & R_u(2) \\ R_u(1) & R_u(0) & R_u(1) \\ R_u(2) & R_u(1) & R_u(0) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \cos(\omega) & \cos(2\omega) \\ \cos(\omega) & 1 & \cos(\omega) \\ \cos(2\omega) & \cos(\omega) & 1 \end{bmatrix}}_{\triangleq \bar{R}} \frac{A^2}{2}$$

$$|\bar{R}| = \begin{vmatrix} 1 & \cos(\omega) & \cos(2\omega) \\ \cos(\omega) & 1 & \cos(\omega) \\ \cos(2\omega) & \cos(\omega) & 1 \end{vmatrix} \begin{matrix} -\cos(\omega)\cdot * \\ * \\ -\cos(\omega)\cdot * \end{matrix} =$$

$$= \begin{vmatrix} 1-\cos^2(\omega) & 0 & \cos(2\omega)-\cos^2(\omega) \\ \cos(\omega) & 1 & \cos(\omega) \\ \cos(2\omega)-\cos^2(\omega) & 0 & 1-\cos^2(\omega) \end{vmatrix} =$$

$$= \begin{vmatrix} 1-\cos^2(\omega) & \cos(2\omega)-\cos^2(\omega) \\ \cos(2\omega)-\cos^2(\omega) & 1-\cos^2(\omega) \end{vmatrix} =$$

$$= \Big/ \cos(2\omega) = 2\cos^2(\omega)-1 \Big/ =$$

$$= \begin{vmatrix} 1-\cos^2(\omega) & 2\cos^2(\omega)-1-\cos^2(\omega) \\ 2\cos^2(\omega)-1-\cos^2(\omega) & 1-\cos^2(\omega) \end{vmatrix} =$$

$$= \begin{vmatrix} 1-\cos^2(\omega) & \cos^2(\omega)-1 \\ \cos^2(\omega)-1 & 1-\cos^2(\omega) \end{vmatrix} =$$

$$= (1-\cos^2(\omega))^2 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$$

$\therefore$ R är singulär $\Rightarrow$

systemet ej identifierbart!