Question 1

Consider the problem: Find \( u(x) \) such that

\[-(a(x)u'(x))' + c(x)u(x) = f(x), \quad x \in I = (0, 1),\]

\[u(0) = 0, \quad a(1)u'(1) = \alpha,\]

where \( a(x) \geq a_0 > 0, \quad c(x) \geq c_0 > 0, \quad \text{and} \quad f(x) \) are given functions.

(a) Derive the variational form. [P]

(b) Let \( 0 = x_0 < x_1 < \cdots < x_N = 1 \) be a discretization of \( I \). Derive the finite element method using continuous piecewise linear basis functions. Present the resulting linear system of equations. [P]

(c) The entries in the load-vector are often assembled by using some quadrature rule. Give an example and write down the resulting formula in the present context. [P]

(d) Suppose that \( \alpha = 0 \). Prove that there is a constant \( C \) such that

\[\|u\|_{H^1(I)} \leq C\|f\|_{L^2(I)}\]

in terms of the \( H^1(I) \)-norm \( \|v\|_{H^1(I)} := \|v\|_{L^2(I)} + \|v'\|_{L^2(I)} \). [H]

Question 2

A reaction-diffusion equation in 2D is given by \( u_t = \kappa \Delta u - \lambda u \) (where \( \kappa \) and \( \lambda \) are positive constants) and is posed in some smooth domain \( \Omega \) with homogeneous Neumann boundary conditions and given initial data \( u = u_0 \) for \( t = 0 \).

(a) Formulate in continuous time a finite element method using standard linear basis functions on a triangulation \( K \) of \( \Omega \). Use the trapezoidal rule for the time discretization and take care in formulating the equations that need to be solved in each step. [P]

(b) At a certain time \( T \), the error \( e \) in the \( L^2(\Omega) \)-norm is computed by comparing to a known analytical solution. By varying the spatial discretization \( h := \max_K \mathcal{h}_K \) and the time-step \( k \) the results in the table are obtained. Estimate the missing entries. [P]

<table>
<thead>
<tr>
<th>( h \times 10^{-3} \times 10 )</th>
<th>( (h, k) )</th>
<th>( (h/2, k) )</th>
<th>( (h, k/2) )</th>
<th>( (h/2, k/2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.874</td>
<td>?</td>
<td>0.857</td>
<td>?</td>
<td></td>
</tr>
</tbody>
</table>

(c) For the analytical solution \( u \), prove decay in the \( L^2(\Omega) \)-norm so that \( \|u\| \leq \|u_0\| \). Prove that the same result holds true for the fully discrete solution \( U_n \) from (a). [H]

Question 3

(a) Write down the (triangle) \( T \)-matrix for the mesh in Figure 1. There are two triangles that are of particularly poor quality. Suggest your own fix to this and indicate how the operation affects the \( T \)-matrix. [P]

(b) Refine all triangles which contain node 8 uniformly once such that the resulting mesh remains a valid triangulation. Explain briefly how you reason and draw the final mesh. [H]
(c) Suppose that the standard linear basis functions are employed in a 3-dimensional domain $\Omega$ to discretize some well-posed second order time-independent PDE. Assume that all tetrahedra are uniformly refined several times (in such a refinement each tetrahedra becomes 8 smaller ones). What is the asymptotic relation between the error measured in the $L^2(\Omega)$-norm and the number of tetrahedra $N_K$? Hint: For a mesh refined $k$ times, what is $h\,N_K$? \[P\]

(d) Write a mini-essay where you reflect over how finite element software typically works. Indicate the workflow when solving a PDE over some geometry. Try to write a maximum of $\sim 10$ sentences. \[P\]

Question 4

Consider the PDE $-\Delta u = f$ in a domain $\Omega \subset \mathbb{R}^2$ with homogeneous Dirichlet boundary conditions.

(a) Derive the variational and finite element formulations (including the discrete set of equations to be solved). Assume a given triangulation $K := \cup K$ is available, where $K$ are the triangles of the mesh. \[P\]

(b) State and prove a version of Galerkin orthogonality for this problem. Subsequently derive a best approximation result and explain the label “best approximation”. \[H\]

(c) Derive the estimate $\|\nabla(u - U)\|^2_{L^2(\Omega)} \leq \text{const.} \times \sum_{K \in K} h_K^2 \|D^2u\|^2_{L^2(K)}$, where $h_K = \text{diam}(K)$ and where $U$ is the FEM solution. Use without proof whatever interpolation estimates you believe you need, for example $\|\nabla(u - \pi u)\|^2_{L^2(K)} \leq Ch_K^2 \|D^2u\|^2_{L^2(K)}$. \[H\]

(d) A typical refinement criterion in adaptive FEM-codes is to refine element $i$ whenever $R_i(U) \geq \eta \max_i R_i(U),$ where $R_i(U)$ is the a posteriori error estimate for element $i$. Explain the effect when the parameter $\eta$ is varied. \[H\]

Question 5

A simple version of the 2D wave equation is $u_{tt} = \Delta u$ for $t > 0$ in some domain $\Omega$ with homogeneous Neumann conditions on $\partial\Omega$. Assume that suitable initial data $u = u_0$ and $u_t = v_0$ for $t = 0$ are available.

(a) Derive a semi-discrete (time-continuous) finite element method using standard linear basis functions on a triangulation $K$ of $\Omega$. Then discretize time by Heun’s method and formulate the equations that need to be solved in each step. For the ODE $y' = f(y)$, Heun’s method is $y_{n+1} = y_n + (K_1 + K_2)/2$ with $K_1 = kf(y_n)$, $K_2 = kf(y_n + K_1)$, and $k$ a discrete time-step. \[P\]

(b) Prove that the corresponding steady-state problem $u_{tt} = 0$ is indefinite. Explain! \[H\]