UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing

Exam in Finite element methods 2011-04-26

- *Time:* $08^{00} - 13^{00}$. *Tools:* Pocket calculator, Beta Mathematics Handbook.
- This is an exam *without points*; each problem is graded separately with respect to the learning objectives the problem targets. Problems are marked according to the level of the objective: [P] = goal required to pass, [H] = goal for higher grades.
- All your answers must be well argued and calculations shall be demonstrated in detail. *Solutions that are not complete can still be of value if they include some correct thoughts.*

Question 1

Consider the problem: Find $u(x)$ such that

$$
\begin{aligned}
-(a(x)u'(x))' + c(x)u(x) &= f(x), \quad x \in I = (0,1), \\
u(0) = 0, \quad a(1)u'(1) &= \alpha,
\end{aligned}
$$

where $a(x) \geq a_0 > 0$, $c(x) \geq c_0 > 0$, and $f(x)$ are given functions.

(a) Derive the variational form. [P]

(b) Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be a discretization of I. Derive the finite element method using continuous piecewise linear basis functions. Present the resulting linear system of equations. [P]

(c) The entries in the load-vector are often assembled by using some quadrature rule. Give an example and write down the resulting formula in the present context. [P]

(d) Suppose that $\alpha = 0$. Prove that there is a constant C such that $\|u\|_{H^1(I)} \leq C\|f\|_{L^2(I)}$ in terms of the $H^1(I)$-norm $\|v\|^2_{H^1(I)} := \|v\|^2_{L^2(I)} + \|v'\|^2_{L^2(I)}$. [H]

Question 2

A *reaction-diffusion* equation in 2D is given by $u_t = \kappa\Delta u - \lambda u$ (where κ and λ are positive constants) and is posed in some smooth domain Ω with homogeneous Neumann boundary conditions and given initial data $u = u_0$ for $t = 0$.

(a) Formulate in continuous time a finite element method using standard linear basis functions on a triangulation $\mathcal{K}$ of Ω. Use the trapezoidal rule for the time discretization and take care in formulating the equations that need to be solved in each step. [P]

(b) At a certain time T, the error e in the $L^2(\Omega)$-norm is computed by comparing to a known analytical solution. By varying the spatial discretization $h := \max_K h_K$ and the time-step k the results in the table are obtained. Estimate the missing entries. [P]

	(h, k)	$(h/2, k)$	$(h, k/2)$	$(h/2, k/2)$
$e \times 10^3$	0.874	?	0.857	?

(c) For the analytical solution u, prove decay in the $L^2(\Omega)$-norm so that $\|u\| \leq \|u_0\|$. Prove that the same result holds true for the fully discrete solution U_n from (a). [H]

Question 3

(a) Write down the (triangle) T-matrix for the mesh in Figure 1. There are two triangles that are of particularly poor quality. Suggest your own fix to this and indicate how the operation affects the T-matrix. [P]

(b) Refine all triangles which contain node 8 uniformly once such that the resulting mesh remains a valid triangulation. Explain briefly how you reason and draw the final mesh. [H]

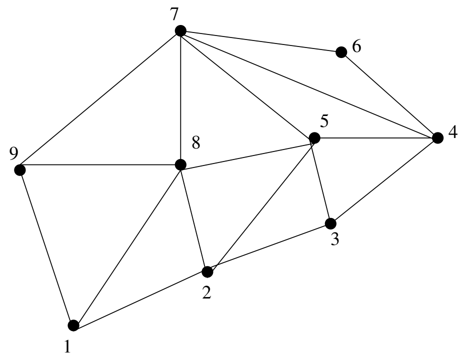

Figure 1: Sample 2D triangulation with the node numbering indicated.

(c) Suppose that the standard linear basis functions are employed in a *3-dimensional* domain Ω to discretize some well-posed second order time-*in*dependent PDE. Assume that all tetrahedra are uniformly refined several times (in such a refinement each tetrahedra becomes 8 smaller ones). What is the asymptotic relation between the error measured in the $L^2(\Omega)$-norm and the number of tetrahedra N_K? *Hint:* For a mesh refined k times, what is h? N_K? [P]

(d) Write a mini-essay where you reflect over how finite element software typically works. Indicate the work-flow when solving a PDE over some geometry. Try to write a maximum of $\sim$ 10 sentences. [P]

Question 4

Consider the PDE $-\Delta u = f$ in a domain $\Omega \subset \mathbf{R}^2$ with homogeneous Dirichlet boundary conditions.

(a) Derive the variational and finite element formulations (including the discrete set of equations to be solved). Assume a given triangulation $\mathcal{K} := \cup K$ is available, where K are the triangles of the mesh. [P]

(b) State and prove a version of *Galerkin orthogonality* for this problem. Subsequently derive a *best approximation result* and explain the label "best approximation". [H]

(c) Derive the estimate $\|\nabla(u - U)\|^2_{L^2(\Omega)} \leq \text{const.} \times \sum_{K \in \mathcal{K}} h_K^2 \|D^2 u\|^2_{L^2(K)}$, where $h_K = \text{diam}(K)$ and where U is the FEM solution. Use without proof whatever interpolation estimates you believe you need, for example $\|\nabla(u - \pi u)\|^2_{L^2(K)} \leq C h_K^2 \|D^2 u\|^2_{L^2(K)}$. [H]

(d) A typical refinement criterion in adaptive FEM-codes is to refine element i whenever $R_i(U) \geq \eta \max_i R_i(U)$, where $R_i(U)$ is the *a posteriori* error estimate for element i. Explain the effect when the parameter η is varied. [H]

Question 5

A simple version of the 2D wave equation is $u_{tt} = \Delta u$ for $t > 0$ in some domain Ω with homogeneous Neumann conditions on $\partial\Omega$. Assume that suitable initial data $u = u_0$ and $u_t = v_0$ for $t = 0$ are available.

(a) Derive a semi-discrete (time-continuous) finite element method using standard linear basis functions on a triangulation $\mathcal{K}$ of Ω. Then discretize time by *Heun's method* and formulate the equations that need to be solved in each step. For the ODE $y' = f(y)$, Heun's method is $y_{n+1} = y_n + (K_1 + K_2)/2$ with $K_1 = kf(y_n)$, $K_2 = kf(y_n + K_1)$, and k a discrete time-step. [P]

(b) Prove that the corresponding steady-state problem $u_{tt} = 0$ is indefinite. Explain! [H]

Good luck!
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