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Department of Information Technology
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Solutions to exam in Finite element methods 2011-04-26

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Question 1

(a) Multiplying the PDE by a test-function v and integrating by parts using the BCs we readily get $-v(1)\alpha + (v', au') + (v, cu) = (v, f)$. Define $V^0 = \{w;\ \|w\| + \|w'\| < \infty, w(0) = 0\}$ where the usual $L^2(I)$-norm is understood. The variational form (VF) is "Find $u \in V^0$ s.t. $-v(1)\alpha + (v', au') + (v, cu) = (v, f)$ for $\forall v \in V^0$".

(b) Defining

$$V_{h,0} = \{w;\ w \text{ continuous and piecewise linear on the given discretization}, w(0) = 0\}$$

we see that a basis of $V_{h,0}$ is $\{\varphi_i\}_{i=1}^N$ (where φ_i is the hat-function centered at x_i). The FEM is just "Find $U \in V_{h,0}$ s.t. $-v(1)\alpha + (v', aU') + (v, cU) = (v, f)$ for $\forall v \in V_{h,0}$". Using the ansatz $U = \sum_{j=1}^N \xi_j \varphi_j$ we get the fully discrete FEM $(A + M)\xi = b + d$ in terms of $A_{ij} = (\varphi_i', a(x)\varphi_j')$, $M_{ij} = (\varphi_i, c(x)\varphi_j)$, $b_i = (\varphi_i, f)$, and $d_N = \alpha$ (and zero elsewhere), where in all cases $i, j = 1 \ldots N$.

(c) An example is the trapezoidal rule:

$$\begin{aligned} b_i = \int_{x_{i-1}}^{x_{i+1}} \varphi_i(x) f(x)\,dx &\approx \frac{x_i - x_{i-1}}{2} [\varphi_i(x_{i-1}) f(x_{i-1}) + \varphi_i(x_i) f(x_i)] + \\ &\quad \frac{x_{i+1} - x_i}{2} [\varphi_i(x_i) f(x_i) + \varphi_i(x_{i+1}) f(x_{i+1})] \\ &= \frac{x_i - x_{i-1}}{2} f(x_i) + \frac{x_{i+1} - x_i}{2} f(x_i). \end{aligned}$$

Note that quadratures can not generally be applied blindly across the whole interval $[x_{i-1}, x_{i+1}]$ since φ_i is not continuously differentiable at x_i.

(d) Using (VF) with $v = u$ and Cauchy-Schwartz we get $\|u\|\|f\| \geq (u, f) = (u', au') + (u, cu) \geq a_0\|u'\|^2 + c_0\|u\|^2 \geq C^{-1}(\|u'\|^2 + \|u\|^2)$ for some constant $C > 0$. Using $\|u\| = (\|u\|^2)^{1/2} \leq (\|u\|^2 + \|u'\|^2)^{1/2}$ on the left yields the desired result.

Question 2

(a) Multiplying with a test-function $v \in V := \{w;\ \|w\| + \|\nabla w\| < \infty\}$ we get using Green's formula and the BCs that $(v, u_t) = -\kappa(\nabla v, \nabla u) - \lambda(v, u)$ in the $L^2(\Omega)$-inner product and induced norm (defined also for vector-valued functions by $(F, G) := \int_\Omega F \cdot G\,dx$). Given the triangulation $\mathcal{K}$, define $V_h := \{w;\ w \text{ piecewise linear and continuous on } \mathcal{K}\}$. Using the standard basis $\{\varphi_j\}_{j=1}^N$ for V_h we get the discrete FEM $M\xi_t = -(\kappa A + \lambda M)\xi$ for $t > 0$. The initial data can be obtained from a projection as $M\xi_0 = d$ (solved once). Here $M_{ij} = (\varphi_i, \varphi_j)$, $A_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$, and $d_i = (\varphi_i, u_0)$.

Using the trapezoidal rule results in the linear system of equations $(M + k/2\,B)\,\xi_{n+1} = (M - k/2B)\,\xi_n$, in terms of $B = \kappa A + \lambda\,M$, time-step k, and $n = 0, 1, \ldots$

(b) Since the trapezoidal rule is 2nd order accurate, the $L^2(\Omega)$-error is expected to behave as $O(k^2) + O(h^2)$. Using the ansatz *error* $= c_1k^2 + c_2h^2$ we get from the table the two equations

$$\begin{aligned} c_1k^2 + c_2h^2 &= 0.874 \times 10^{-3}, \\ c_1(k/2)^2 + c_2h^2 &= 0.857 \times 10^{-3}. \end{aligned}$$

Solving for the missing entries we get

$$c_1 k^2 + c_2 (h/2)^2 = 0.2355 \times 10^{-3},$$
$$c_1 (k/2)^2 + c_2 (h/2)^2 = 0.2185 \times 10^{-3}.$$

Hence the spatial error dominates at this resolution.

(c) Multiplying the PDE with the exact solution u and integrating using Green's formula we get $(u, u_t) = d/dt\,\|u\|^2/2 = -\kappa\|\nabla u\|^2 - \lambda\|u\|^2 \le 0$ which proves the first assertion. For the discrete case it is convenient to write the method as $M(\xi_{n+1}-\xi_n)/k = -B/2(\xi_{n+1}+\xi_n)$ and multiply with $(\xi_{n+1}+\xi_n)^T$. We get $(\|U_{n+1}\|^2 - \|U_n\|^2)/k = -\kappa/2\|\nabla U_{n+1} + \nabla U_n\|^2 - \lambda/2\|U_{n+1} + U_n\|^2 \le 0$. Finally, the energy method applied to the initial data yields $\|U_0\|^2 = (U_0, u_0) \le \|U_0\|\|u_0\|$ so that $\|U_n\| \le \cdots \|U_0\| \le \|u_0\|$.

Question 3

(a) A T-matrix that will do is (note the convention of numbering nodes in a triangle counterclockwise)

$$T = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 4 & 4 & 5 & 8 \\ 8 & 2 & 5 & 3 & 4 & 6 & 7 & 7 & 7 \\ 9 & 8 & 8 & 5 & 5 & 7 & 5 & 8 & 9 \end{bmatrix}.$$

Triangles 4-6-7 and 4-7-5 are of poor quality since they are very flat. If an extra node is inserted between nodes 5 and 6 (see Figure 1), the T-matrix becomes

$$T = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 & (4) & (n) & (5) & (n) & 4 & 5 & 8 \\ 8 & 2 & 5 & 3 & 4 & (6) & (6) & (4) & (7) & 7 & 7 & 7 \\ 9 & 8 & 8 & 5 & 5 & (n) & (7) & (n) & (5) & 5 & 8 & 9 \end{bmatrix},$$

with n the new node. Another fix is to simply remove the edge 4-7 and insert an edge 5-6 instead.

(b) See Figure 1.

(c) The error behaves as $O(h^2)$ with h a measure of the maximum diameter of all tetrahedra. If a mesh is refined uniformly k times, we should have $h \propto 2^{-k}$ since each refinement splits all edges in two. Likewise, $N_K \propto 8^k = 2^{3k}$ since each tetraheder becomes 8 smaller ones. Hence $\mathit{error} \sim h^2 \propto 2^{-2k} = (2^{3k})^{-2/3} \propto N_K^{-2/3}$. A shorter alternative solution is to note that in 3D, $h \sim N_K^{-1/3}$. Hence $\mathit{error} \sim h^2 \sim N_K^{-2/3}$.

(d) One often starts with some kind of CAD/CAM-work to specify the geometry over which the PDE is to be defined. Next the PDE itself is specified by giving terms in a predetermined form or specifying some kind of "general" PDE. Also, boundary conditions must be supplied such that the PDE is well-posed. After that the geometry should be discretized with a proper triangulation (not always so easy!). With a mesh in hand the FEM can be assembled numerically using quadratures and a first tentative solution is obtained (might take a while if the mesh is large). The solution can be post-processed using some kind of visualization tools to judge if the mesh needs to be refined. In that case, the process starts over from generating the mesh anew. — Alternatively, an adaptive solver of some kind can be used.

Question 4

(a) Homogeneous Dirichlet boundary conditions means that $u|_{\partial\Omega} = 0$. Multiplying the PDE by a test-function v satisfying those conditions and integrating using Green's formula we readily get $(\nabla v, \nabla u) = (v, f)$. Define $V_0 = \{w;\ \|w\| + \|\nabla w\| < \infty,\ w|_{\partial\Omega} = 0\}$. The variational form (VF) is "Find $u \in V_0$ s.t. $(\nabla v, \nabla u) = (v, f)$ for $\forall v \in V_0$". We define $V_{h,0} = \{w;\ w$ piecewise linear and continuous on $\mathcal{K}$, $u|_{\partial\Omega} = 0\}$ and let $\{\varphi_i\}_{i=1}^N$ be a basis of $V_{h,0}$ consisting of the standard tent-functions on $\mathcal{K}$ (with no nodes on the boundary).

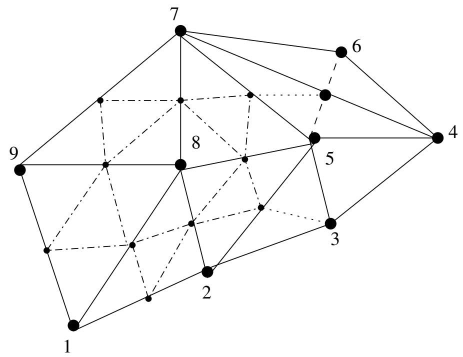

Figure 1: The mesh after inserting a new node between nodes 5 and 6 and adding two edges (dashed). Then the triangles containing node 8 are uniformly refined by adding 10 new nodes (small circles) and 5 edges (dash-dot), ensuring also a valid triangulation by connecting hanging nodes appropriately (dotted). The new nodes have not yet received any numbers.

The FEM is just "Find $U \in V_{h,0}$ s.t. $(\nabla v, \nabla U) = (v, f)$ for $\forall v \in V_{h,0}$". Using the ansatz $U = \sum_{j=1}^{N} \varphi_j \xi_j$ we get the fully discrete FEM $A\xi = b$ in terms of $A_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$, and $b_i = (\varphi_i, f)$ for $i,\, j = 1 \ldots N$.

(b) By subtracting (FEM) from (VF) we get the Galerkin orthogonality $(\nabla v, \nabla u - \nabla U) = 0$ for $\forall v \in V_{h,0}$. Defining the error to be $e = u - U$ we have that $\|\nabla e\|^2 = (\nabla u - \nabla U, \nabla u - \nabla U) = (\nabla u - \nabla v, \nabla u - \nabla U) + (\nabla v - \nabla U, \nabla u - \nabla U)$. For $v, U \in V_{h,0}$ we therefore get using Galerkin orthogonality that $\|\nabla e\|^2 = (\nabla u - \nabla v, \nabla u - \nabla U) \le \|\nabla u - \nabla v\| \|\nabla e\|$ from Cauchy-Schwartz. Hence $\|\nabla e\| \le \|\nabla u - \nabla v\|$ for $\forall v \in V_{h,0}$. That is, in the sense of the energy norm, U is a best approximation in $V_{h,0}$.

(c) From Galerkin orthogonality we get $\|\nabla e\|^2 = (\nabla e, \nabla e - \nabla(\pi e)) = \sum_i (\nabla e, \nabla e - \nabla(\pi e))_i$ where the sum is over elements $i = 1 \ldots N$ and the inner product is the one over element K_i. From Cauchy-Schwartz we get $\|\nabla e\|^2 \le \sum_i \|\nabla e\|_i \|\nabla e - \nabla(\pi e)\|_i \le C \sum_i \|\nabla e\|_i h_i \|D^2 u\|_i \le C\|\nabla e\| \left(\sum_i h_i^2 \|D^2 u\|_i^2\right)^{1/2}$ using the suggested interpolation estimate and the (discrete) Cauchy-Schwarz inequality. Rearranging things a bit we get the stated estimate.

(d) The factor η controls how generously elements are added to the mesh. For $\eta \to 0$ we will have a high confidence in the error estimate and add many elements for each iteration. When $\eta \to 1$, instead only the worst offending elements will be subdivided. Hence one will have to continue the iterations longer (although each iteration will be cheaper).

Question 5

(a) Multiplying with a test-function $v \in V := \{w;\ \|w\| + \|\nabla w\| < \infty\}$ and integrating using Green's formula and the homogeneous Neumann conditions we get $(v, u_{tt}) = -(\nabla v, \nabla u)$ in terms of the $L^2(\Omega)$-inner product (defined as usual also for vector-valued functions by $(F, G) := \int_\Omega F \cdot G\, dx$). Define $V_h = \{w;\ w \text{ piecewise linear and continuous on } \mathcal{K}\}$ and let $\{\varphi_j\}_{j=1}^N$ be a basis. A semi-discrete FEM is now $M\xi_{tt} = -A\xi$ in terms of the FE-solution $U = \sum_j \varphi_j \xi_j(t)$, $M_{ij} = (\varphi_i, \varphi_j)$, and $A_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$. The initial data can be obtained by projection as $M\xi(0) = c$ and $M\xi_t(0) = d$ with $c_i = (\varphi_i, u_0)$ and $d_i = (\varphi_i, v_0)$. To discretize in time, define the variable $\eta = \xi_t$ so that $M\eta(0) = d$ and

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \xi_t \\ \eta_t \end{bmatrix} = \begin{bmatrix} 0 & M \\ -A & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Heun's method can now be applied;

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} K_1^{(1)} \\ K_1^{(2)} \end{bmatrix} = k \begin{bmatrix} 0 & M \\ -A & 0 \end{bmatrix} \begin{bmatrix} \xi_n \\ \eta_n \end{bmatrix},$$

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} K_2^{(1)} \\ K_2^{(2)} \end{bmatrix} = k \begin{bmatrix} 0 & M \\ -A & 0 \end{bmatrix} \begin{bmatrix} \xi_n + K_1^{(1)} \\ \eta_n + K_1^{(2)} \end{bmatrix},$$

and finally,

$$\begin{bmatrix} \xi_{n+1} \\ \eta_{n+1} \end{bmatrix} = \begin{bmatrix} \xi_n + (K_1^{(1)} + K_2^{(1)})/2 \\ \eta_n + (K_1^{(2)} + K_2^{(2)})/2 \end{bmatrix},$$

in terms of the time-step k. The initial data are given as before by $M\xi_0 = c$ and $M\eta_0 = d$.

(b) Suppose that u is a steady-state solution so that $\Delta u = 0$ with homogeneous Neumann BCs. Then $u +$ const. is also a solution. With an isolating boundary condition the steady-state solution evidently depends on the initial data.