UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing
Exam in Finite element methods 2012-12-14

- *Time:* $08^{00} - 13^{00}$. *Tools:* Pocket calculator, Beta Mathematics Handbook.
- This is an exam *without points*; each problem is graded separately with respect to the learning objectives the problem targets. Problems are marked according to the level of the objective: [P] = goal required to pass, [H] = goal for higher grades.
- All your answers must be well argued and calculations shall be demonstrated in detail. *Solutions that are not complete can still be of value if they include some correct thoughts.*

Question 1

Consider the problem: Find $u(x)$ such that

$$-(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x), \quad x \in I = (0,1),$$
$$u(0) = u(1) = \alpha,$$

where $a(x) \geq a_0 > 0$, and where $b(x)$, $c(x)$, and $f(x)$ are given bounded functions.

(a) Derive the variational form. [P]

(b) Let $0 = x_0 < \cdots < x_N = 1$ be a discretization of I. Present a FEM using continuous piecewise linear basis functions over this mesh. State the resulting system of equations. [P]

(c) Give an explicit expression for $\varphi_1(x)$ when the mesh is defined by the nodes $[0, 1/2, 1]$. [P]

(d) Suppose that $a(x) = \varepsilon > 0$, $b(x) = 1$, and that $\alpha = c(x) = 0$. Derive an *a priori* stability estimate for the solution to the variational form. What happens when $\varepsilon \to 0$ and how can this be a problem to the finite element method? [H]

Question 2

For certain applications one may seek the solution to Maxwell's equations in terms of an *electric scalar potential* u. The time-dependent potential formulation thus reads in two space dimensions, $-(\Delta u_t + c\Delta u) = f$ in some smooth convex domain Ω. The boundary conditions are homogeneous Neumann for $x \in \Gamma_N$ and homogeneous Dirichlet for $x \in \Gamma_D$ (such that $\partial\Omega = \Gamma_D \cup \Gamma_N$). Initial data for $t = 0$ is given by $u = u_0(x)$. Reasonable assumptions are that the constant $c > 0$ and that $f(x)$ is absolutely bounded.

(a) Formulate a fully discrete FEM for this problem using backward Euler to discretize time and a triangulation $\mathcal{K}$ of Ω and the standard linear basis functions to discretize space. Take care in formulating the equations that need to be solved in each step. *Hint:* If you want to you can start by discretizing time after making the substitution $w = -\Delta u$. Next you can formulate in already discrete time the finite element method. [P]

(b) For these types of problems it is often not the potential u that is of interest, but rather the *electric field* $E := -\nabla u$. Assume that at some time T, the error e *in the electric field* E has been estimated to satisfy $\|e\| \approx 10^{-2}$ in the $L^2(\Omega)$-norm. Given that the backward Euler method is first order accurate in the time-step k, and that the spatial discretization is parametrized by $h := \max_K h_K$, estimate how much smaller (h, k) needs to be in order to reach an $L^2(\Omega)$-error $\approx 10^{-4}$ *in the electric field* E. [P]

(c) Assume that the source term $f = 0$. For the analytical solution u, prove decay in the energy-norm so that $\|\nabla u\|_{L^2(\Omega)} \leq \|\nabla u_0\|_{L^2(\Omega)}$. Prove that the same result holds true for the fully discrete solution from (a). [H]

Question 3

(a) Indicate the sparsity pattern for the mass-matrix associated with the mesh in Figure 1.

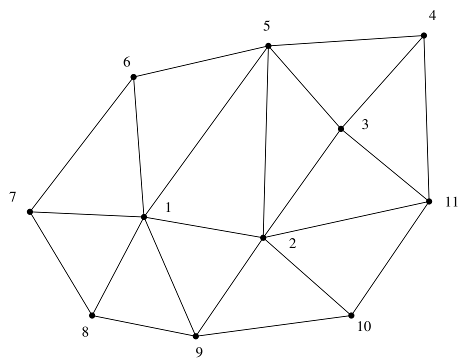

Figure 1: A 2D triangulation.

Suppose we solve $-\Delta u = f$ with Dirichlet boundary conditions on all boundaries. What is the sparsity pattern of the stiffness matrix in this case? [P]

(b) Refine all triangles which contain node 3 uniformly once while ensuring that the mesh remains a valid triangulation. Draw the resulting mesh and also explain the meaning of the term 'hanging node' — preferably by giving an example. [H]

(c) Suppose that a finite element software appears to work poorly for a certain problem. Discuss a few reasons for this behavior. Also, how could we detect this? [P]

(d) Finite element software typically has to assemble sparse matrices for the equations to be solved. Explain in your own words using a maximum of ~ 10 sentences how this process works. You can use a simple model example for the discussion if you want. [P]

Question 4

Consider the task of projecting a function $f \in L^2(\Omega)$ on discrete subspaces V_h, V_H where $V_H \subset V_h \subset L^2(\Omega)$, and where $\Omega \subset \mathbf{R}^2$ is a polygonal domain.

(a) Derive the variational formulations for the projections $P_h : L^2(\Omega) \to V_h$ and $P_H : L^2(\Omega) \to V_H$. Next, assume a triangulation $\mathcal{K} := \cup K$ is available, where K are the triangles of the mesh and suppose that $V_H = \{w;\ w \text{ piecewise linear and continuous on } \mathcal{K}\}$. Present the equations that need to be solved when forming P_H using the standard basis. [P]

(b) Let $u_h = P_h f$ and $u_H = P_H f$ and define $e = u_h - u_H$. Prove that $e \perp V_H$. Also show that $\|e\|_{L^2(\Omega)} \leq \|u_h - v\|_{L^2(\Omega)}$, $\forall v \in V_H$. What does this means in terms of approximating functions in V_h with those in V_H? [H]

(c) Derive the estimate $\|u_h - u_H\|^2_{L^2(\Omega)} \leq CH^2 \times \|Du_h\|^2_{L^2(\Omega)}$, where C is a constant, $H_K \equiv \operatorname{diam}(K)$, and $H = \max_K H_K$. Use without proof whatever interpolation estimates you believe you need, for example $\|g - \pi_H g\|^2_{L^2(K)} \leq CH_K^2 \|Dg\|^2_{L^2(K)}$. [H]

(d) Prove that $\|u_h - u_H\|^2 = \|u_h\|^2 - \|u_H\|^2$. [H]

Question 5

A version of the 2D wave equation is $u_{tt} = \nabla \cdot c(x)^2 \nabla u$ for $t > 0$ in some domain Ω with homogeneous Dirichlet conditions on $\partial\Omega$. Assume that $c(x) \geq c_0 > 0$ and that suitable initial data $u = u_0$ and $u_t = v_0$ for $t = 0$ are available.

(a) Derive a semi-discrete (time-continuous) finite element method using standard linear basis functions on a triangulation $\mathcal{K}$ of Ω. Then discretize time by the trapezoidal rule and formulate the equations that need to be solved in each step. [P]

(b) Prove that the energy $E = \|u_t\|^2_{L^2(\Omega)} + \|c(x)\nabla u\|^2_{L^2(\Omega)}$ is constant in time. [H]

Good luck!
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