Fictitious Domain Methods and Topology Optimization

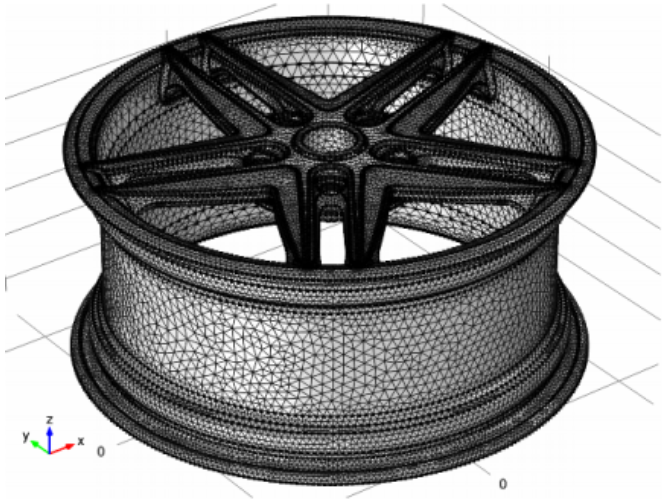
Martin Berggren

UMIT research lab
Department of Computing Science
Umeå University

April 11, 2014

Geometric flexibility

- The geometric flexibility of the FEM an important reason for its success
- The FEM works well with **unstructured meshes**
- Unstructured meshes can be generated fast and automatically in many cases



Unstructured mesh (from Comsol Multiphysics manual)

Typical FEM procedure

Three distinctive steps:

- **Preprocessing:** Geometry definition (often from CAD) + generation of body-fitted mesh
- **Analysis:** Assembly of system matrices, solution of the algebraic equations
- **Postprocessing:** Visualization, qualitative and quantitative analysis of the solution

Limitations of the typical procedure

The clear separation between preprocessing and analysis sometimes problematic:

- The computational domain changes with time, e.g. fluid–structure interaction
- The geometry is a part of the **solution** of the problem, e.g. design optimization, free boundary problems

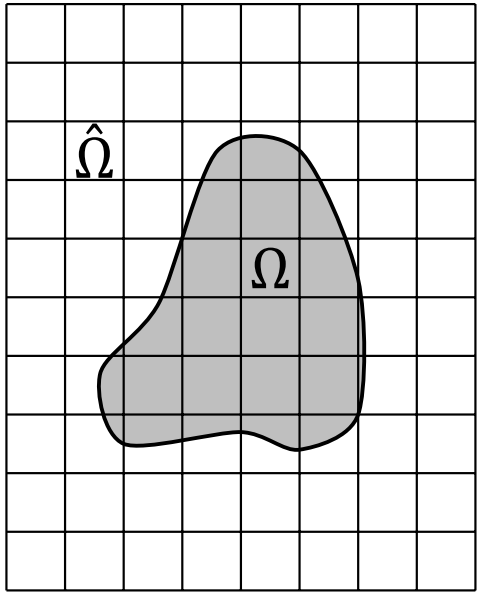
In those cases, can be an advantage to **integrate** geometry handling and analysis

Fictitious domain methods (also called *domain embedding methods*) accomplish this

Fictitious domain methods

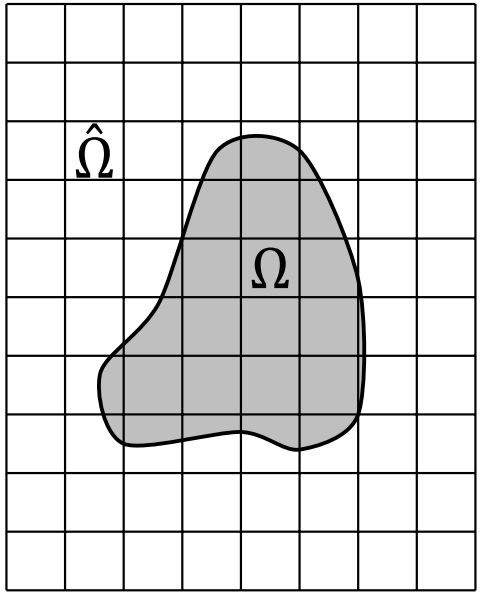
Basic idea:

- A large (often square) computational domain $\hat{\Omega}$ covers "real", physical domain Ω
- The computational domain is meshed using a **fixed** (often regularly structured) mesh
- The equations are solved on the larger, computational domain
- The boundary conditions on the "real" domain becomes interior conditions in the computational domain
- The point with it all: easier to change the interior conditions than to remesh and interpolate on the new mesh



Fictitious domain methods

- The boundary conditions at the interior boundary have to be "forced" somehow.
- Two basic ways for this:
  - Through extra or modified **boundary integrals** in the variational form
  - Through extra or modified **volume integrals** in the variational form



Today: a basic (but very useful!) **volume integral** version.

Natural vs. essential boundary conditions

The three standard boundary conditions for second-order elliptic boundary-value problems:
(i) Dirichlet, (ii) Neumann, and (iii) Robin conditions

$$-\Delta u = f \qquad \text{in } \Omega \tag{1a}$$

$$\text{(Dirichlet)} \qquad u = 0 \qquad \text{on } \partial\Omega \tag{1b}$$

$$\text{(Neumann)} \qquad \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega \tag{1c}$$

$$\text{(Robin)} \qquad \alpha u + \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega \tag{1d}$$

Note: Case (1c) requires $\int_\Omega f \, dV = 0$ for consistency.

Variational forms

Multiply equation (1a) with a test function $v \in H^1(\Omega)$[†], integrate, and use Green's formula:

$$\int_\Omega v f \, dV = -\int_\Omega v \Delta u \, dV = -\int_{\partial\Omega} v \frac{\partial u}{\partial n} \, dS + \int_\Omega \nabla v \cdot \nabla u \, dV$$

$$\left[\text{Dirichlet BC: let } v \equiv 0 \text{ on } \partial\Omega\right] = \int_\Omega \nabla v \cdot \nabla u \, dV$$

$$\left[\text{Neumann BC: } \partial u / \partial n \equiv 0 \text{ on } \partial\Omega\right] = \int_\Omega \nabla v \cdot \nabla u \, dV$$

$$\left[\text{Robin BC: } \partial u / \partial n = -\alpha u \text{ on } \partial\Omega\right] = \int_{\partial\Omega} \alpha v u \, dS + \int_\Omega \nabla v \cdot \nabla u \, dV$$

[†] $H^1(\Omega)$: the space of square-integrable functions with square-integrable derivatives

Variational forms

H_0^1: the subspace of $H^1(\Omega)$ functions that vanish on the boundary

PDE with
Dirichlet boundary condition
Var. form:
Essential boundary condition

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla v \cdot \nabla u \, dV = \int_\Omega v f \, dV \quad \forall v \in H_0^1(\Omega)$$

PDE with
Neumann boundary condition
Var. form:
Natural boundary condition

Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla v \cdot \nabla u \, dV = \int_\Omega v f \, dV \quad \forall v \in H^1(\Omega)$$

PDE with
Robin boundary condition
Var. form:
Natural boundary condition
+ bdy. integral

Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla v \cdot \nabla u \, dV + \int_{\partial\Omega} \alpha v u \, dS = \int_\Omega v f \, dV \quad \forall v \in H^1(\Omega)$$

FE approximation & implementation

FEM: variational form solved in space $V_h \subset H^1(\Omega)$ of continuous, piecewise polynomial functions on a mesh with N nodes.

Expanding in a nodal basis:

$$u_h(x) = \sum_{i=1}^{N} u_i \phi_i(x) \qquad \text{for } u_h \in V_h \subset H^1(\Omega)$$

Essential boundary conditions are enforced **explicitly** on u_h.

Standard approach for $u_h = 0$ on the boundary, remove index for nodes on boundaries $I_{\text{int}} \subset \{1, \dots, N\}$

$$u_h(x) = \sum_{i \in I_{\text{int}}} u_i \phi_i(x) \qquad \text{for } u_h \in V_{h,0} \subset H_0^1(\Omega)$$

FE approximation & implementation

In contrast to essential boundary conditions, the natural conditions coming from Neumann boundary condition are **invisible** in the variational form!

$$\int_\Omega \nabla v_h \cdot \nabla u_h \, dV = \int_\Omega v_h f \, dV \quad \forall v_h \in V_h$$

Suggests a fictitious domain approach!

The "material distribution" approach

K_n: the nth element in a mesh with totally M elements. For Neumann boundary conditions, the variational form can be written

$$\sum_{n=1}^{M} \int_{K_n} \nabla v_h \cdot \nabla u_h \, dV = \sum_{n=1}^{M} \int_{K_n} v_h f \, dV \quad \forall v_h \in V_h$$

Let $\Omega \subset \hat{\Omega}$. Define the **element material indicator**

$$\rho = \begin{cases} 1 & \text{if } K_n \subset \Omega, \\ 0 & \text{otherwise } (K_n \subset \hat{\Omega} \setminus \Omega) \end{cases}$$

and extend the variational form into $\hat{\Omega}$, meshed with $\hat{M} > M$ elements:

$$\sum_{n=1}^{\hat{M}} \int_{K_n} \rho \nabla v_h \cdot \nabla u_h \, dV = \sum_{n=1}^{\hat{M}} \int_{K_n} \rho v_h f \, dV \quad \forall v_h \in V_h$$

or, equivalently

$$\int_{\hat{\Omega}} \rho \nabla v_h \cdot \nabla u_h \, dV = \int_{\hat{\Omega}} \rho v_h f \, dV \quad \forall v_h \in V_h$$

The material distribution approach

Solution u_h undefined in $\hat{\Omega} \setminus \Omega$; stiffness matrix becomes **singular**. Cure: use modified material indicator:

$$\rho = \begin{cases} 1 & \text{if } K_n \subset \Omega, \\ \epsilon & \text{otherwise } (K_n \subset \hat{\Omega} \setminus \Omega) \end{cases}$$

($\epsilon \ll 1$, say $\epsilon = 10^{-6}$)

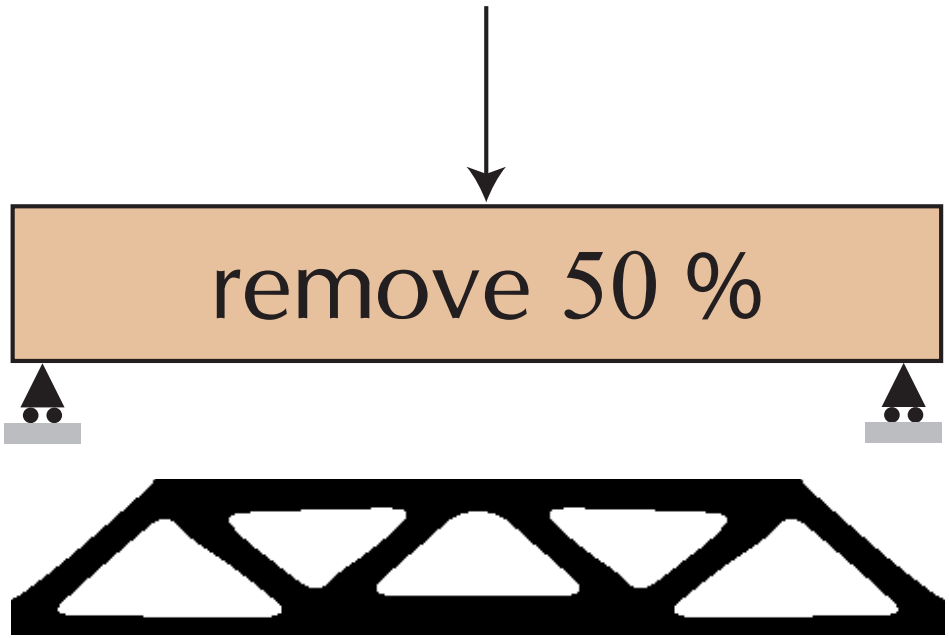
- Fixed mesh: the geometry is now **defined** by the values of function ρ
- An image (pixel) based geometry representation
- Very fine mesh needed for accurate geometry representation

Material distribution optimization

- The element material indicator ρ has often a natural physical interpretation: local heat conductivity, thickness in depth direction, density of material, etc.
- **Material distribution optimization:** use the values of ρ as decision variables in optimization
- Material distribution optimization perhaps the premiere application of this particular fictitious domain approach!

Example 1 (linear elasticity)

- Half of the material is cut out to obtain a structure as stiff as possible
- A so-called **topology optimization problem**: how the body is connected (the number of holes) is not given a priori



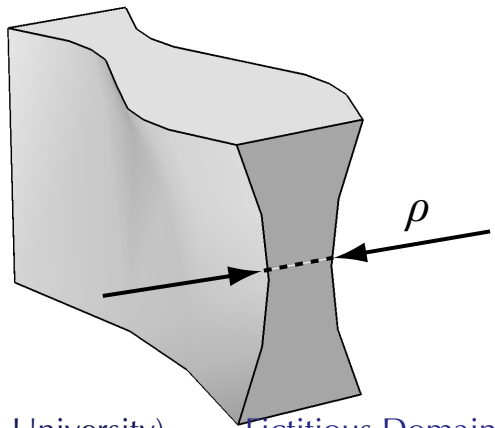
The problem is to divide a **ground structure** ($\hat{\Omega}$) into a **material region** (Ω) and a **non-material (hole) region** ($\omega = \hat{\Omega} \setminus \Omega$)

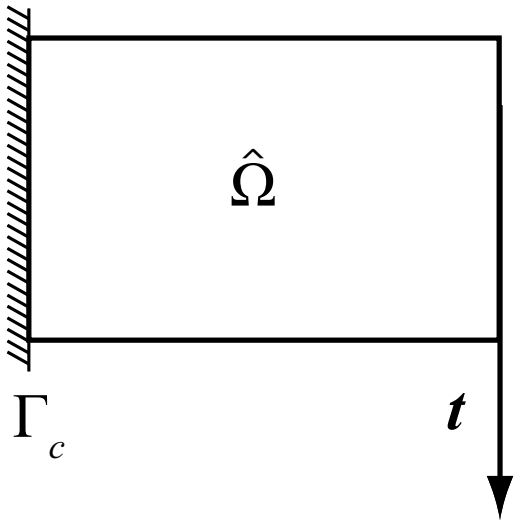
Example 2 (the variable thickness sheet)

- A cantilever beam (sv. *konsolbalk*) is clamped to a wall and vertically loaded at the other side
- The beam consists of a thin sheet of material of variable thickness $0 \leq \rho \leq 1$
- Find the stiffest beam subject to

$$\int_{\hat{\Omega}} \rho \, dV \leq \gamma \int_{\hat{\Omega}} dV$$

for a given $\gamma \in (0, 1)$





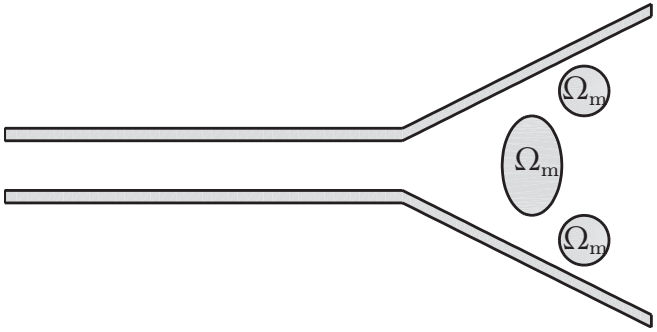
Results for $\gamma = 0.4$

Material distribution optimization for elasticity

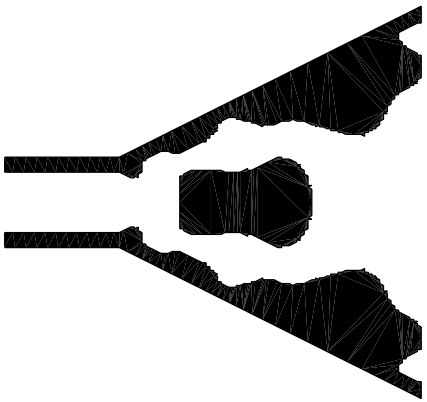
- First numerical demonstration by Bendsøe & Kikuchi (1988).
- Intensely developed since then
- Used for advanced component design, particularly in the car and aeronautical industries
- Three applications of topology optimization carried out on the Airbus A380 aircraft is estimated to have contributed to weight savings in the order of 1000 kg per aircraft

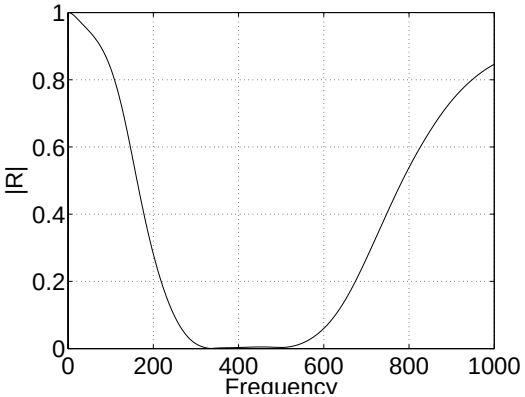
Example 3 (acoustical "labyrinth")

Fill the inside of a funnel-shaped device in order to maximize the transmitted sound energy.
From Wadbro & Berggren (2006):

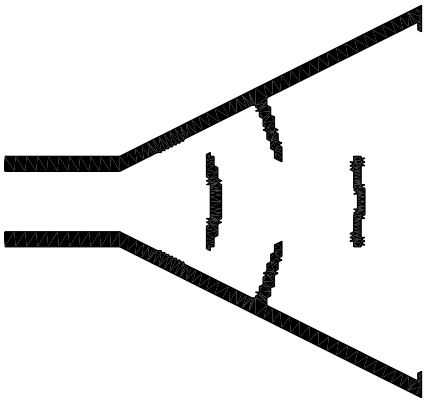


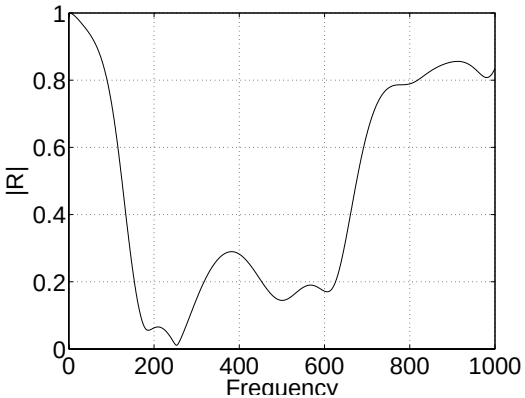
Optimized for frequencies 400, 410, ..., 500 Hz





Optimized for frequencies 150, 160, ..., 300 Hz

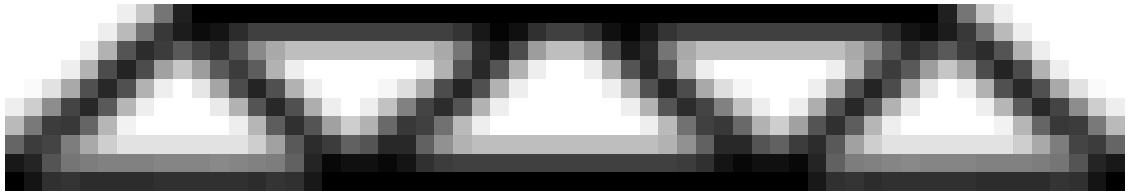




Material distribution optimization with FEM

Conceptually, all three examples use a similar strategy:

- The material indicator function ρ is used as decision variables in the optimization algorithm
- With a crude mesh, the pixel image representation of the geometry is noticeable:



Design variable $\rho \in [0, 1]$: ρ represents thickness in normal direction

Design variable $\rho \in \{ 0, 1 \}$: ρ represents absence/presence of material

Material distribution optimization with FEM

- For the optimization, we introduce an **objective function** J, a numerical performance measure. E. g. the elastic stiffness (example 1 & 2), or the acoustic transmission properties (example 3)
- The objective function is extremized (maximized or minimized) under constraints, e. g. $0 \leq \rho \leq 1$
- We use **gradient-based** optimization method. Needs to calculate $\partial J / \partial \rho_i$, where ρ_i is the design value in element i.
- For the example problems above, gradients are easy to obtain by post processing of the numerical solution
- Iterative procedure: guess ρ in all elements, calculate solution, calculate derivates, update ρ, etc…

The acoustical optimization problem: details

Outline:

1. Acoustics modeling
2. Material modeling
3. The optimization problem
4. Some additional results

The wave and Helmholtz equations

The **acoustic pressure** (pressure fluctuations) satisfies the wave equation

$$\frac{\partial^2 P}{\partial t^2} - c^2 \Delta P = 0$$

We consider time-harmonic solutions $P(x, y, t) = \mathrm{e}^{\mathrm{i}\omega t} p(x, y)$ at angular frequency ω. The complex-valued amplitude function p then satisfies

$$\mathrm{e}^{\mathrm{i}\omega t} \left(-\omega^2 p - c^2 \Delta p \right) = 0,$$

that is, the **Helmholtz equation**

$$-k^2 p - \Delta p = 0,$$

where $k = \omega / c$, the **wave number**

Plane wave solutions

Functions $p(x, y) = \mathrm{e}^{\pm \mathrm{i}kx}$ satisfy the Helmholtz equation

Plane wave traveling towards **increasing** x:

$$P(x, y, t) = \mathrm{e}^{\mathrm{i}(\omega t - kx)} = \mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{-\mathrm{i}kx}$$

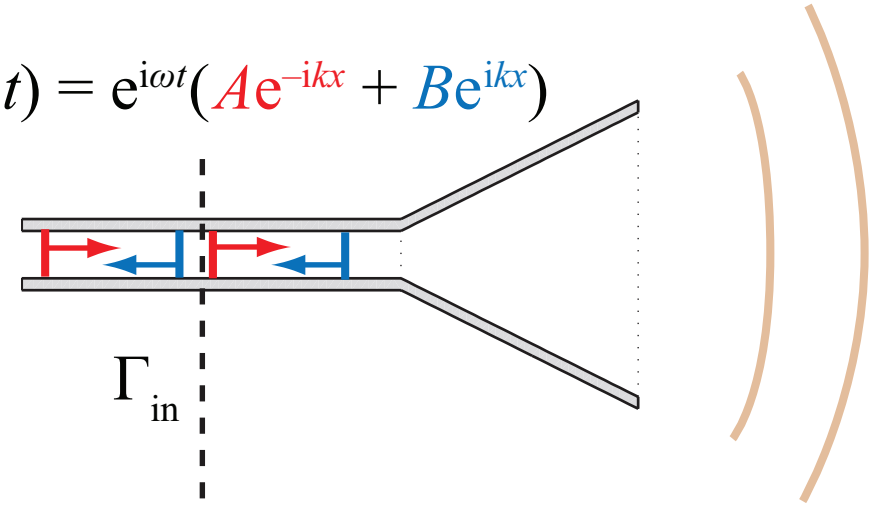
Plane wave traveling towards **decreasing** x:

$$P(x, y, t) = \mathrm{e}^{\mathrm{i}(\omega t + kx)} = \mathrm{e}^{\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}kx}$$

- Plane waves occur in straight narrow channels (acoustic wave guides)
- Assume now that we put a conical-shaped horn at the end of a wave guide

The acoustic horn

$$P(x,y,t) = e^{i\omega t}(Ae^{-ikx} + Be^{ikx})$$



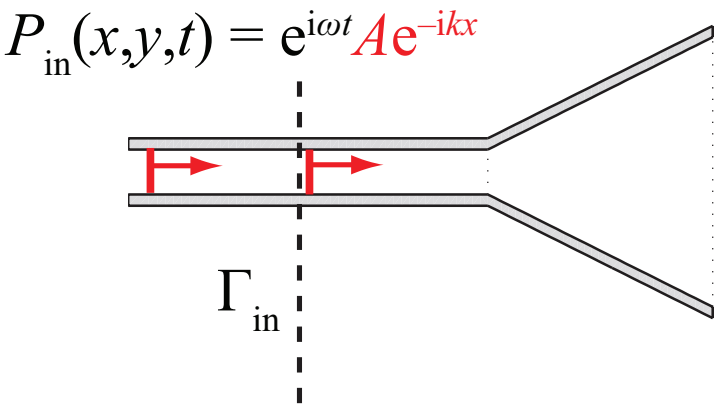
- Parts of plane wave in the wave guide is reflected back, parts is transmitted to free space
- When size of horn $\geq O$(wavelength), the reflection properties will be sensitive to horn shape
  - **Loudspeaker:** want low reflections at **all** transmission frequencies
  - **Brass instrument:** want the waves to be reflected with **zero phase shift** at **certain** pre-specified frequencies to build up standing waves in the feeding wave guide
- The horn directs the sound towards the front of the horn

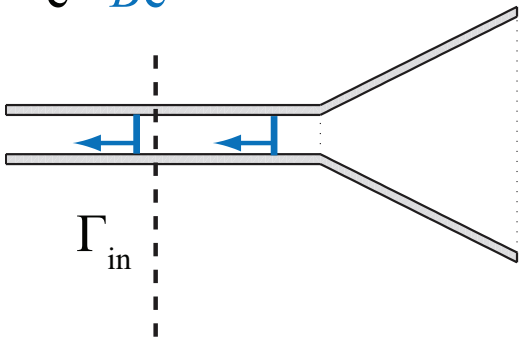
Input/Output perspective

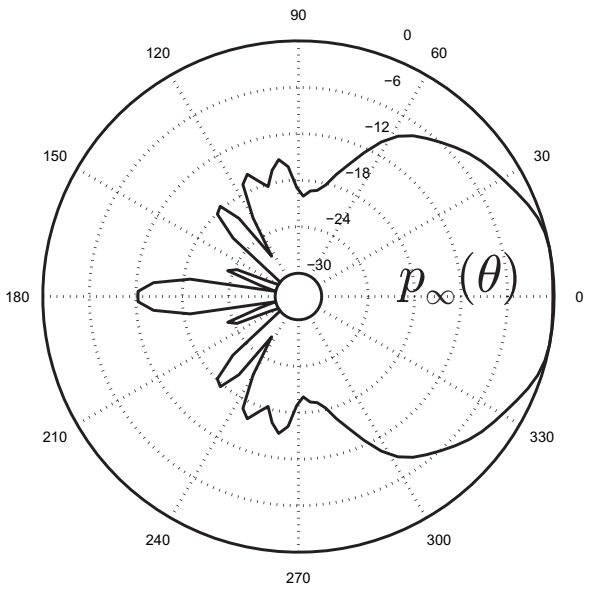
Input

Output

$$P(x,y,t) = e^{i\omega t} B e^{ikx}$$







Inviscid case:

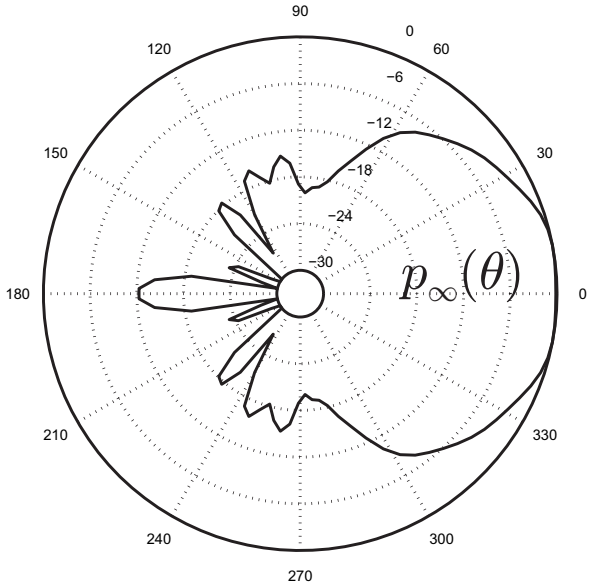
$$|A|^2 = |B|^2 + C \int_0^{2\pi} |p_\infty(\theta)|^2 \, d\theta$$

Loudspeaker optimization objectives

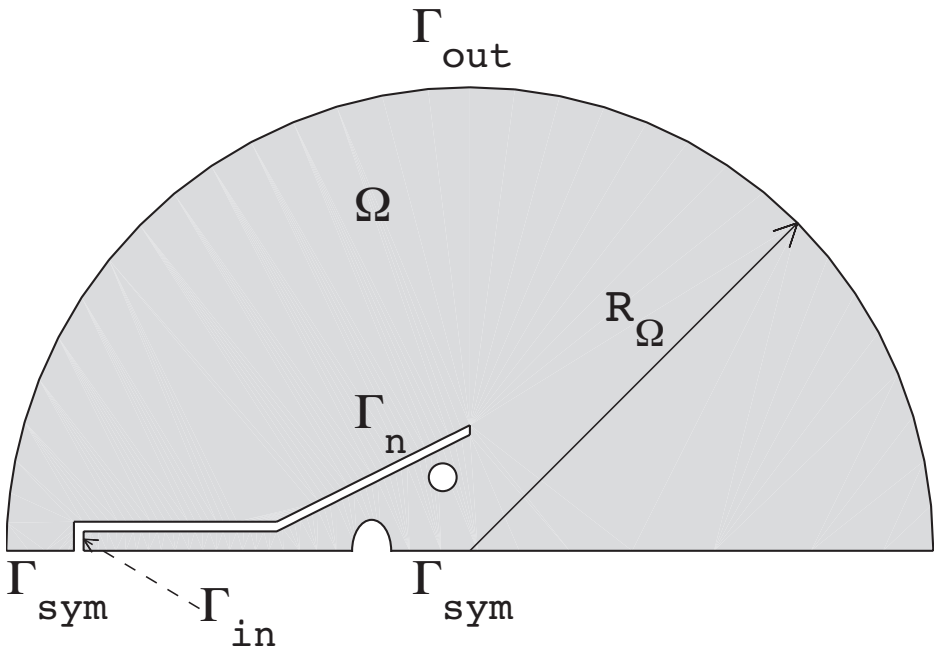
- **Transmission efficiency**: Minimize $|R|^2$, where

$$R = \frac{B}{A} = \frac{p|_{\Gamma_{\text{in}}} - A}{A}$$

  is the *reflection coefficient*. Equivalent to maximizing total exterior output $C \int_0^{2\pi} |p_\infty(\theta)|^2 \, d\theta$
- **Directivity**: The angular distribution of sound in terms of the *far-field pattern* $\theta \mapsto p_\infty(\theta)$



Computational domain



- 2D Helmholtz equation, planar symmetry
- Symmetric up/down with respect to boundary Γ_{sym}
- The infinite domain truncated at Γ_{out}, located at radius R_Ω. Need artificial boundary condition to absorb outgoing waves
- A plane wave source in waveguide, at Γ_{in}, imposing a right-going wave with amplitude A
- All other boundary parts, Γ_n, are **sound hard**. Constitute boundaries to solid materials

Boundary conditions

Condition of symmetry and presence of sound-hard boundaries enforced by

$$\frac{\partial p}{\partial n} = 0 \qquad \text{at } \Gamma_{\text{sym}} \text{ and } \Gamma_n$$

No pressure gradients at walls ⇒ no flow of air through wall.

Only plane waves in wave guide: $p(x, y) = A\mathrm{e}^{-\mathrm{i}kx} + B\mathrm{e}^{\mathrm{i}kx}$. If Γ_{in} is located at $x = 0$, condition

$$\mathrm{i}kp + \frac{\partial p}{\partial n} = 2\mathrm{i}A \qquad \text{at } \Gamma_{\text{in}},$$

sets amplitude A for the right-going wave and absorbs the left-going wave

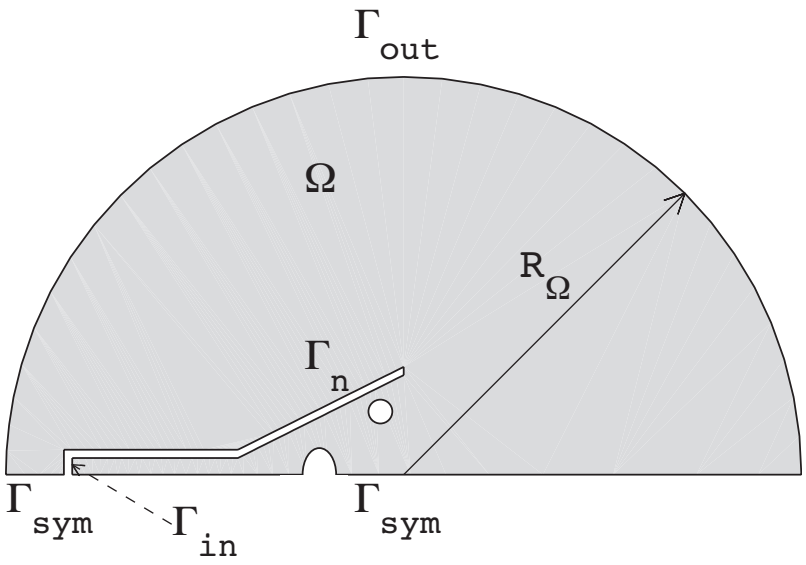
Artificial boundary condition

The so-called first-order Engquist–Majda radiation condition

$$\left(\mathrm{i}k + \frac{1}{2R_\Omega}\right)p + \frac{\partial p}{\partial n} = 0 \qquad \text{at } \Gamma_{\text{out}}$$

absorbs reasonably well outgoing waves that travel perpendicular to Γ_{out}

The boundary-value problem



$$-k^2 p - \Delta p = 0 \quad \text{in } \Omega \tag{2a}$$

$$\left(\mathrm{i}k + \frac{1}{2R_\Omega}\right)p + \frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma_{\text{out}} \tag{2b}$$

$$\mathrm{i}kp + \frac{\partial p}{\partial n} = 2\mathrm{i}A \quad \text{on } \Gamma_{\text{in}} \tag{2c}$$

$$\frac{\partial p}{\partial n} = 0 \quad \text{on } \Gamma_n \tag{2d}$$

Variational form

Recall: for the FE discretization, need to write the equations in **variational form**:

1. Multiply the PDE with a **test function** q and integrate over the computational domain Ω
2. Integrate terms with second derivatives by parts, utilizing **Green's identity**

$$\int_{\partial\Omega} q \frac{\partial p}{\partial n}\, d\Gamma = \int_{\Omega} \nabla q \cdot \nabla p \, d\Omega + \int_{\Omega} q \Delta p \, d\Omega$$

3. Substitute the boundary condition expressions to get rid of $\partial p / \partial n$ terms

Variational form

Multiply (2a) with q and integrate:

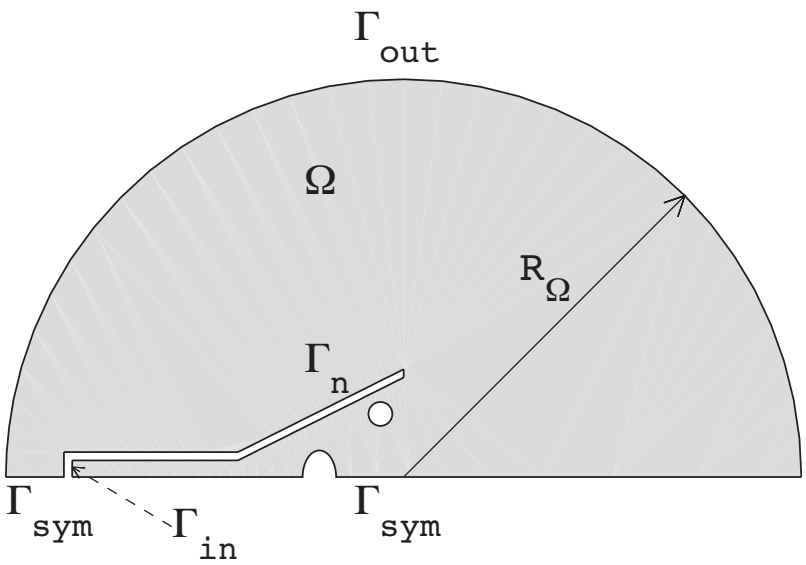
$$
\begin{aligned}
-k^2 \int_\Omega qp\, d\Omega - \int_\Omega q\Delta p\, d\Omega &= [\text{Green's identity}] \\
&= -k^2 \int_\Omega qp\, d\Omega - \int_{\partial\Omega} q\frac{\partial p}{\partial n}\, d\Gamma + \int_\Omega \nabla q \cdot \nabla p\, d\Omega = [\text{Boundary conditions}] \\
&= -k^2 \int_\Omega qp\, d\Omega + \left(\mathrm{i}k + \frac{1}{2R_\Omega}\right)\int_{\Gamma_{\text{out}}} qp\, d\Gamma + \mathrm{i}k \int_{\Gamma_{\text{in}}} qp\, d\Gamma - 2\mathrm{i}\int_{\Gamma_{\text{in}}} qA\, d\Gamma \\
&\quad + \int_\Omega \nabla q \cdot \nabla p\, d\Omega = 0
\end{aligned}
$$

Variational form

Rewriting the expression yields that the solution p satisfies

$$\int_\Omega \nabla q \cdot \nabla p \, d\Omega - k^2 \int_\Omega qp \, d\Omega + \mathrm{i}k \int_{\Gamma_{\text{out}} \cup \Gamma_{\text{in}}} qp \, d\Gamma + \frac{1}{2R_\Omega} \int_{\Gamma_{\text{out}}} qp \, d\Gamma = 2\mathrm{i} \int_{\Gamma_{\text{in}}} qA \, d\Gamma$$

for all test functions q with square-integrable derivatives.



- The boundary conditions on Γ_{in} and Γ_{out} enters as boundary integrals
- As before: the Neumann boundary conditions at sound-hard (and symmetry) boundaries **do not enter in the variational form**!
- In particular, the material regions inside the horn is noticed only through **its absence from the integrals**!

Material modeling

- Extend the domain to $\hat{\Omega} = \Omega \cup \Omega_m$, where Ω_m is the sound-hard material region inside the horn
- Here the material indicator $\rho : \hat{\Omega} \to \mathbb{R}$ signifies **air** or **sound-hard** material

$$\rho(x) = \begin{cases} 1 & \text{if } x \in \Omega \quad \text{(air)} \\ \epsilon & \text{if } x \in \Omega_m \text{ (sound hard)} \end{cases}$$

- Then the variational form can be written

$$\int_{\hat{\Omega}} \rho \nabla q \cdot \nabla p \, d\Omega - k^2 \int_{\hat{\Omega}} \rho q p \, d\Omega + \mathrm{i}k \int_{\Gamma_{\text{out}} \cup \Gamma_{\text{in}}} q p \, d\Gamma + \frac{1}{2R_\Omega} \int_{\Gamma_{\text{out}}} q p \, d\Gamma = 2\mathrm{i} \int_{\Gamma_{\text{out}}} q A \, d\Gamma$$

The function ρ constitutes the **design variables**: the decision variables in the optimization algorithm

Manipulating the design variable

- Optimization using $\rho \in \{\epsilon, 1\}$ numerically difficult.
- Relax the feasible set to the **interval** $\rho \in [\epsilon, 1]$. Allows gradient-based algorithms
- To force either $\rho \approx \epsilon$ or $\rho \approx 1$, penalize the objective:

$$J_\gamma(\rho) = \gamma \int_{\hat{\Omega}} (\rho - \epsilon)(1 - \rho)\, d\Omega + \left| p|_{\Gamma_{\text{in}}} - A \right|^2$$

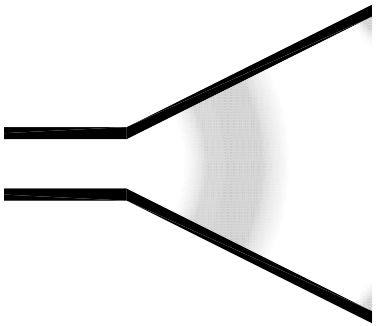
  $\gamma > 0$: penalty parameter
- To avoid microstructures on the element level, define ρ indirectly through a **filter**:

$$\rho(x) = \int_{\mathbb{R}^2} k_R(x - y)\tilde{\rho}(y)\, dV$$

  where integral kernel k_R has support only in a radius R around point x

Relaxation, penalty, filtering, sharpening

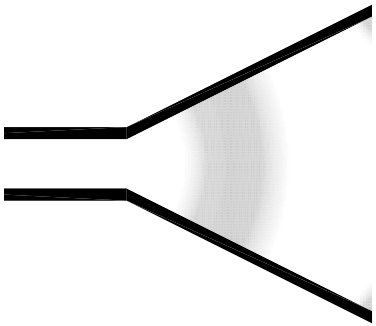
Minimizing reflection at 400 Hz (Wadbro & Berggren 2006)



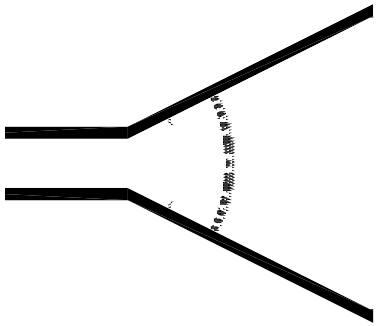
No penalty: $\gamma = 0$; $10^{-3} \leq \rho \leq 1$

Relaxation, penalty, filtering, sharpening

Minimizing reflection at 400 Hz (Wadbro & Berggren 2006)



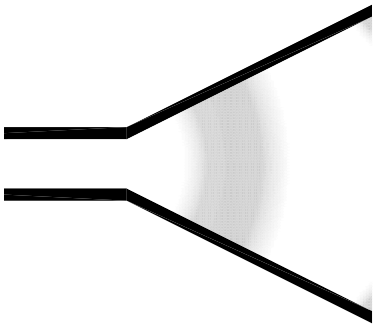
No penalty: $\gamma = 0$; $10^{-3} \leq \rho \leq 1$



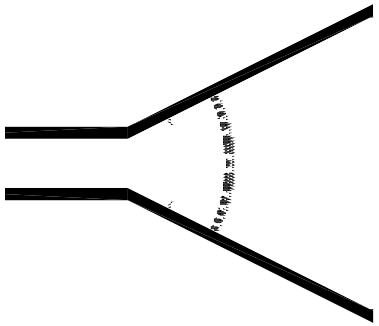
Penalty: $0 = \gamma_0 < \gamma_1 < \gamma_2 < \ldots$. Mesh dependent microstructures appear

Relaxation, penalty, filtering, sharpening

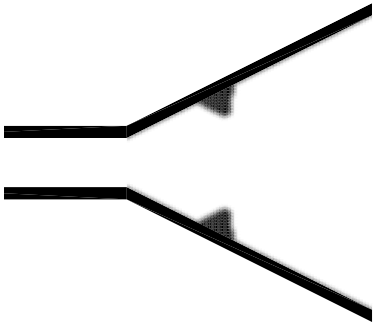
Minimizing reflection at 400 Hz (Wadbro & Berggren 2006)



No penalty: $\gamma = 0$; $10^{-3} \leq \rho \leq 1$



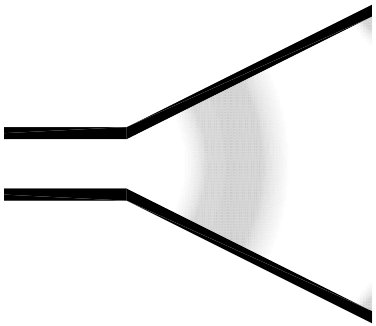
Penalty: $0 = \gamma_0 < \gamma_1 < \gamma_2 < \ldots$. Mesh dependent microstructures appear



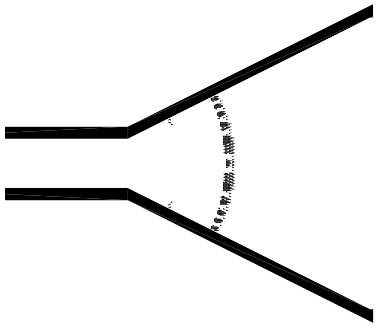
Penalty and filter $\Rightarrow$ mesh independence (note the gray areas of size R!)

Relaxation, penalty, filtering, sharpening

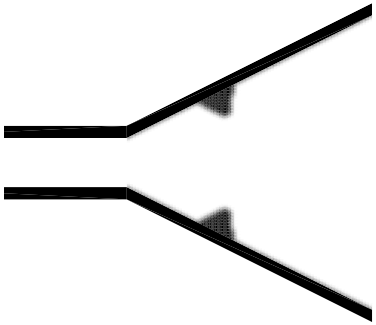
Minimizing reflection at 400 Hz (Wadbro & Berggren 2006)



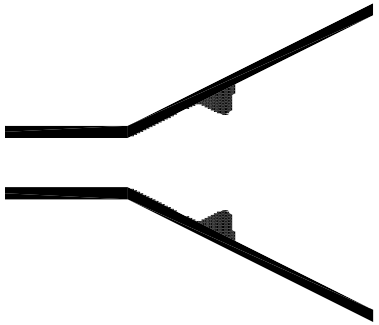
No penalty: $\gamma = 0$; $10^{-3} \leq \rho \leq 1$



Penalty: $0 = \gamma_0 < \gamma_1 < \gamma_2 < \ldots$. Mesh dependent microstructures appear



Penalty and filter ⇒ mesh independence (note the gray areas of size R!)



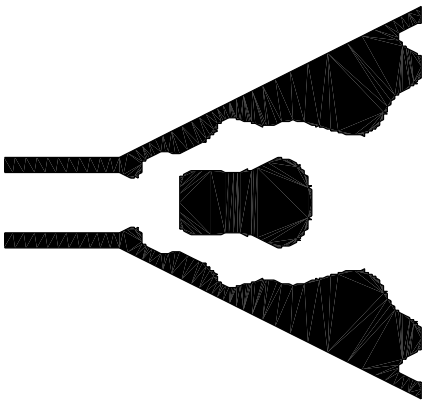
Sharpening: removing the filter and continue the iterations

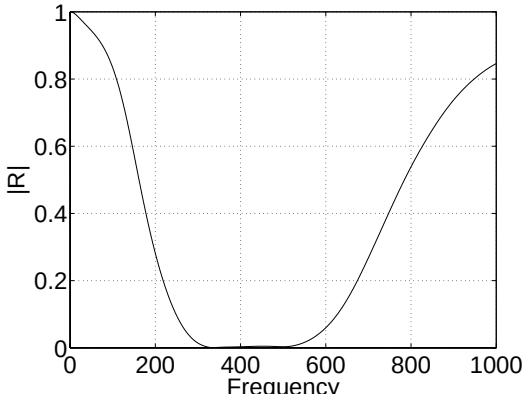
Example 3 revisited

Simultaneous optimization over N_{freq} frequencies:

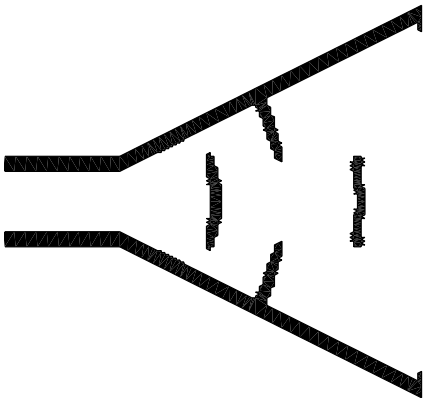
$$J_\gamma(\tilde{\rho}) = \gamma \int_{\hat{\Omega}} (K_R * \tilde{\rho} - \epsilon)(1 - K_R * \tilde{\rho})\, d\Omega + \sum_{n=1}^{N_{\text{freq}}} \left| p_{\omega_n}|_{\Gamma_{\text{in}}} - A \right|^2$$

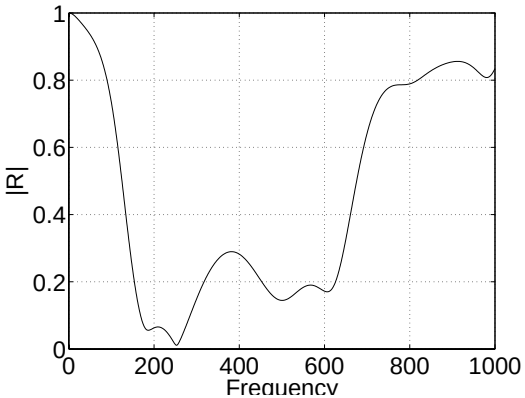
Optimized for frequencies
400, 410, ..., 500 Hz



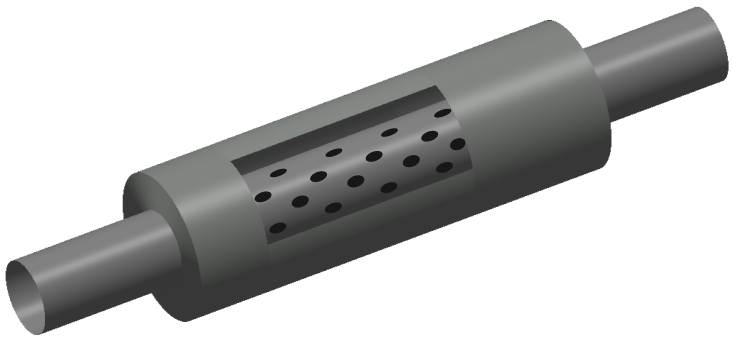


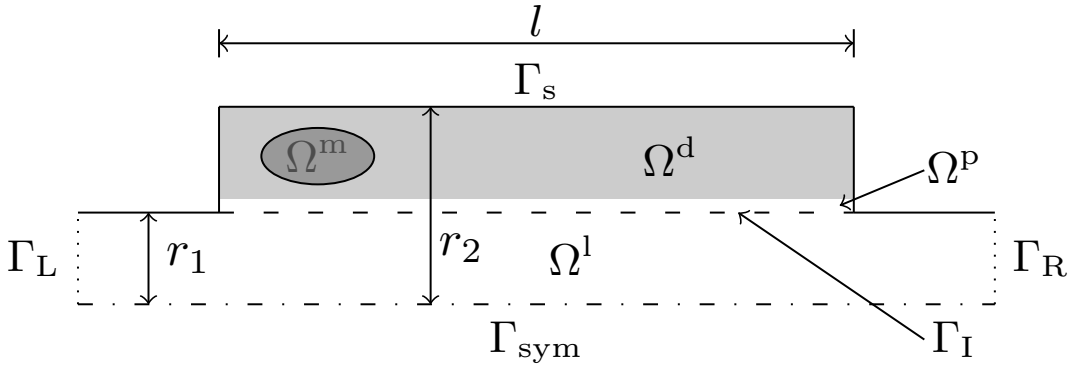
Optimized for frequencies
150, 160, ..., 300 Hz





Example 4: Interior layout of a reactive muffler

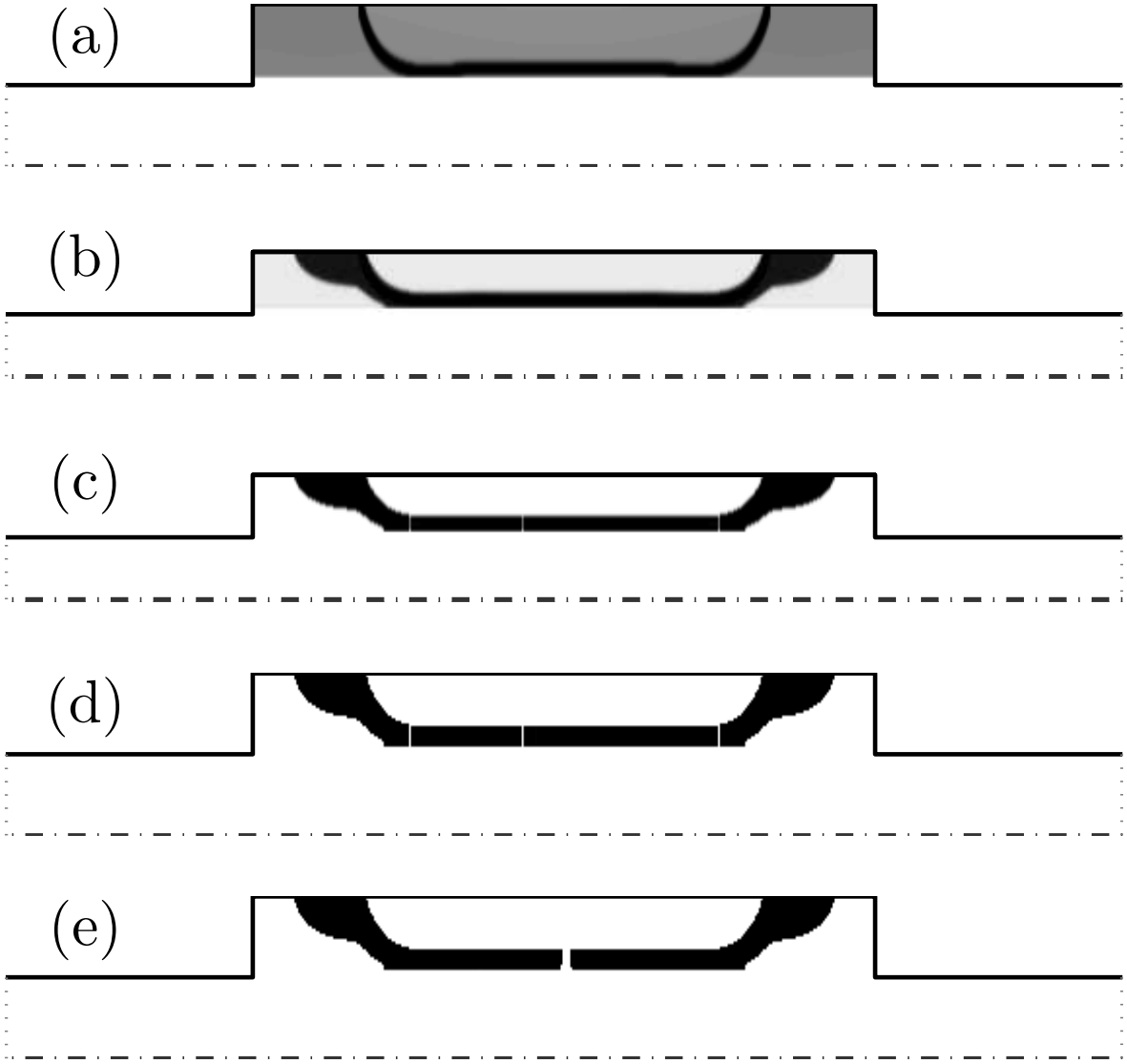




- Cylindrical symmetry
- Perforated pipe possibly running from inlet to outlet
- Placement of sound-hard material in Ω^{d} subject to design
- *Reactive* muffler: opposite objective to the horn: **minimize** the transmission through the muffler!
- With Esubalewe Yedeg and Eddie Wadbro

The making of a Helmholtz resonator

Minimizing transmission at 349 Hz, no pipe



$\gamma = 0$

$\gamma = 1/2$
with filter

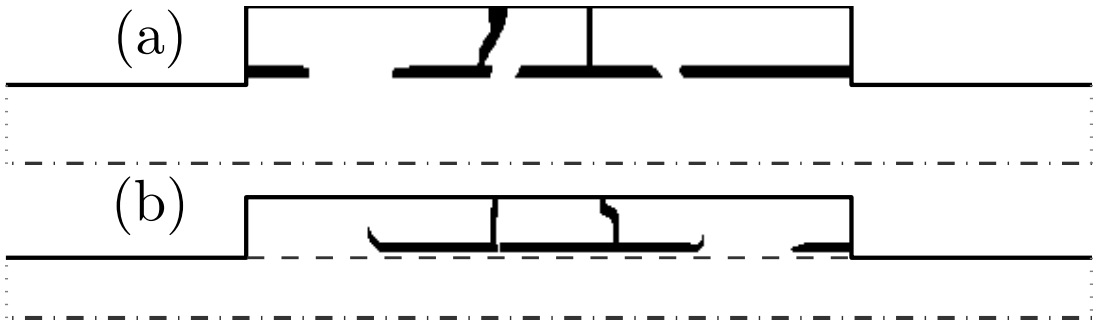
$\gamma = 500$
with filter

$\gamma = 500$
dropping filter

after 2nd
post processing

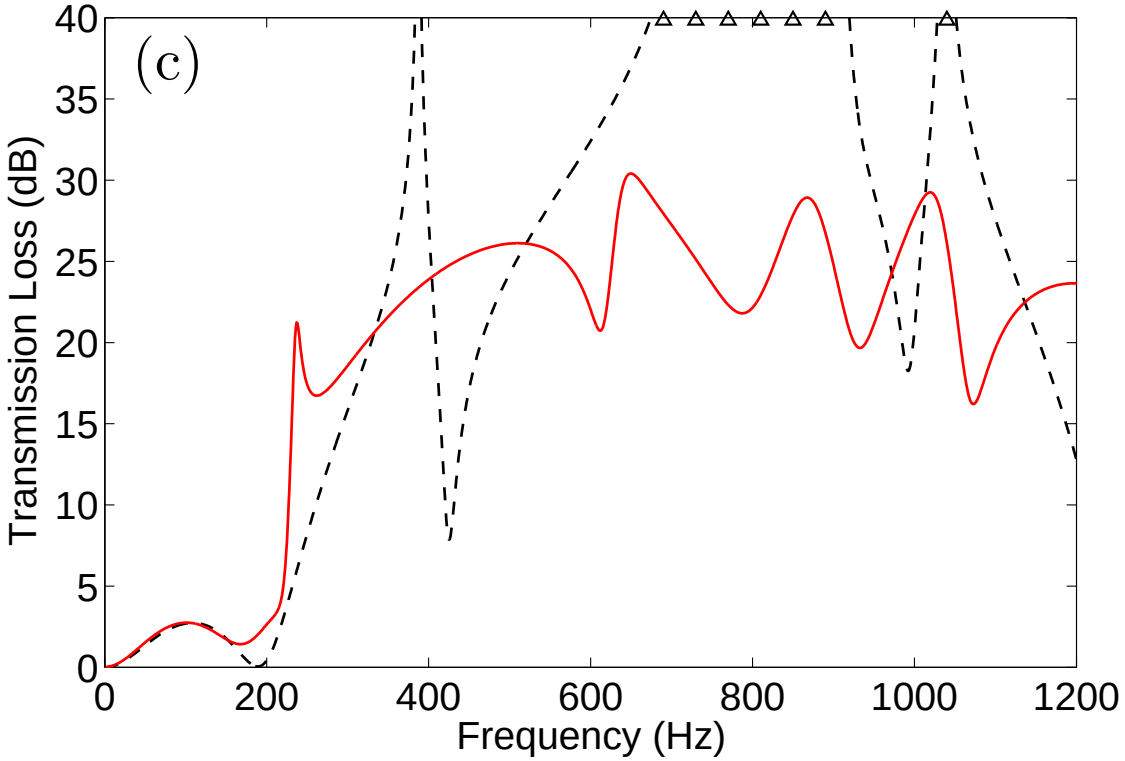
Multifrequency transmission minimization

20 frequencies, 200–1050 Hz. Exponentially spaced



(a) without pipe

(b) with pipe

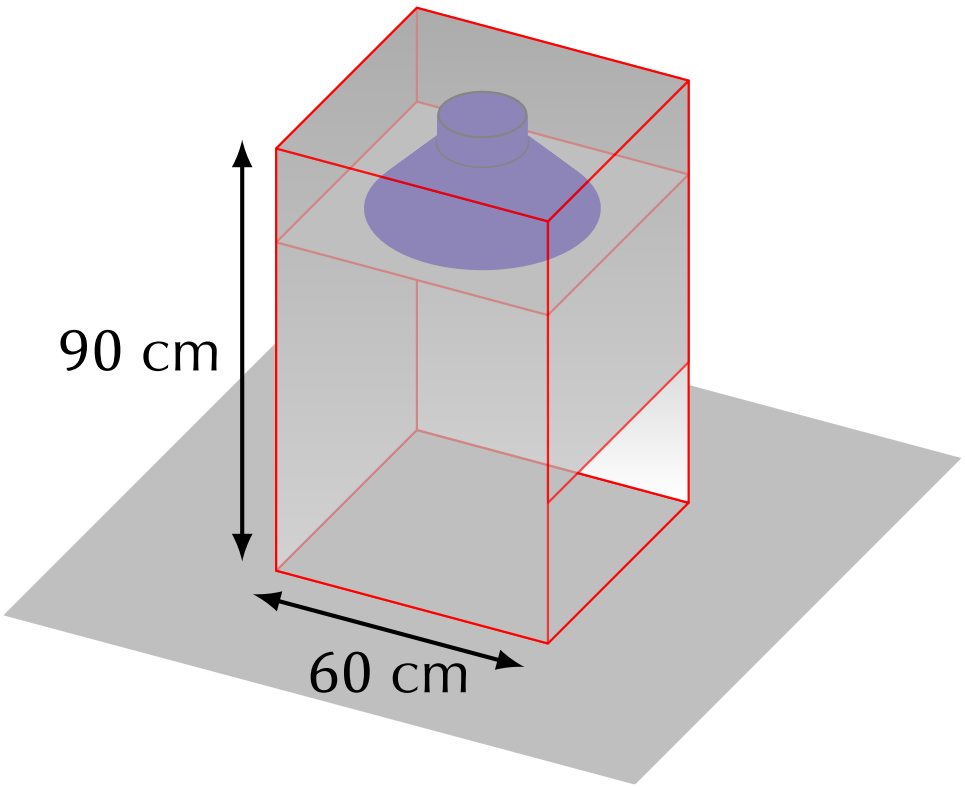


(c) Transmission loss

- - - without pipe

—— with pipe

Example 5: Interior layout of a subwoofer

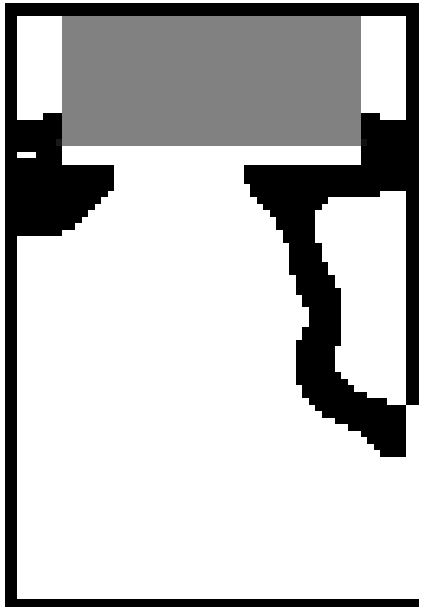


- Design variables:
  $1 \times 1 \times 60$ cm^3 voxels
  air/sound hard material
- Maximize radiated power, fixed input voltage
- Hybrid model:
  - 3D Helmholtz models around driver (FEM) and exterior (BEM)
  - 2D Helmholtz in box
  - Mechanical and electrical circuits for driver

Preliminary results

Work in progress!

Optimized 40–80 Hz



Optimized 80–160 Hz