UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing
Exam in Finite element methods II 2014-06-05

- *Time:* $08^{00} - 13^{00}$. *Tools:* Pocket calculator, Beta Mathematics Handbook.
- This is an exam *without points*; each problem is graded separately with respect to the learning objectives the problem targets. Problems are marked according to the level of the objective: [P] = goal required to pass, [H] = goal for higher grades.
- All your answers must be well argued and calculations shall be demonstrated in detail. *Solutions that are not complete can still be of value if they include some correct thoughts.*

Question 1

Consider the problem in d dimensions,

$$\begin{aligned} \nabla \cdot A\nabla u + b \cdot \nabla u + cu &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

where $A(\bar{x}) = a_{i,j}$ is a symmetric matrix ($a_{i,j}(\bar{x}) = a_{j,i}(\bar{x})$), $b(\bar{x})$ is a vector, and $c(\bar{x})$ a scalar function of $\bar{x} = (x_1, x_2 \ldots, x_d)$ $(i, j = 1, 2, \ldots, d)$.

(a) Derive the variational formulation and give the Galerkin approximating of (1). [P1]

(b) Given that $y^T A(x)y \geq a_0 y^T y$ for all $y \in \mathbb{R}^d$ and $a_0 > 0$, $c - \frac{1}{2}\nabla \cdot b > c_0 > 0$, $f \in L^2(\Omega)$, and $a_{i,j}, b_i, c \in L^\infty(\Omega)$ for all i and j, show that the Lax-Milgram lemma can be applied in order to prove well-posedness. [P2]

(c) Assume that $u \in H^3(\Omega)$ solves the problem in a weak sense. Assume that we have the interpolation estimate [P3]

$$||\nabla(v - \pi v)|| \leq C_\pi h^{\min(m-1,p)} |u|_{H^m(\Omega)}$$

where $\pi : H^m(\Omega) \to V_h$ and p is the order of the Lagrange element in V_h. Prove an *a priori* error estimate of the Galerkin approximation $u_h \in V_h$.

(d) Let $b = 0$, choose $F(v)$ such that (1) is equivalent to a minimization problem [H1]

$$F(u) = \min_{v \in H_0^1(\Omega)} F(v)$$

Prove that the finite element solution u_h is a minimizer of $F(v)$ in V_h.

Question 2

(a) Trilinear Lagrange basis function on a unit cube are spanned by the canonical basis $\{1, r, s, t, rs, rt, st, rst\}$. Let the functionals $L_i(v) = v(N_i)$ for $i = 1, \ldots, 8$ where N_i is the corners in the unit cube be given. Write down a linear system to compute the coefficient for the ith basis function $\phi_i = c_i^1 + c_i^2 r + c_i^3 s + c_i^4 t + c_i^5 rs + c_i^6 rt + c_i^7 st + c_i^8 rst$, $i \in \{1, \ldots, 8\}$, using the defining functionals $L_i(\cdot)$. Explain the set-up for this basis using an illustrative picture. [P4]

(b) An *a priori* error bound using non-conforming finite element ($V_h \not\subset V$) can have the form

$$\|u - u_h\|_h \leq C \left(\inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{w \in V_h \neq 0} \frac{|(f, w) - a_h(u, w)|}{\|w\|_h} \right) \tag{2}$$

where the first term is the approximation error and the second term is the consistency error, for $u \in V$ and where $u_h \in V_h$ solves $a_h(u_h, v) = (f, v)$ for all $v \in V_h$. Assume that $a_h(\cdot, \cdot)$ is coercive in V_h and continuous on $(V + V_h)$ in the norm $\| \cdot \|_h$. Prove (2). *Hint:* start from $\|u - u_h\|_h \leq \inf_{v_h \in V_h} \|u - v_h\|_h + \|v_h - u_h\|_h$ and prove $\|v_h - u_h\|_h \leq C_1 \|u - v_h\|_h + C_1 \sup_{w \in V_h \neq 0} |(f, w) - a_h(u, w)| / \|w\|_h$, where C_1 and C_2 depends on the coercivity and continuity constants. [P4]

Question 3

Consider the problem

$$\begin{aligned} \epsilon\Delta u + b\cdot\nabla u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

in 2D for $f \in L^2(\Omega)$ and $\nabla \cdot b = 0$.

(a) Motivate why stabilization is needed in the case $b = (1,1)$ and $\epsilon \ll 1$. [H3]

(b) Given a triagulation $\mathcal{T}$ of the domain Ω, the GLS reads: find $u_h \in V_h$ such that

$$a_h(u_h, v) = \ell(v) \quad \forall v \in V_h,$$

where

$$\begin{aligned} a_h(u_h, v) &= \epsilon(\nabla u_h, \nabla v) + (b\cdot\nabla u_h, v) + \delta \sum_{T\in\mathcal{T}} (-\epsilon\Delta u_h + b\cdot\nabla u_h, -\epsilon\Delta v + b\cdot\nabla v) \\ \ell(v) &= (f, v) + \delta\sum_{T\in\mathcal{T}} (f, -\epsilon\Delta v + b\cdot\nabla v). \end{aligned}$$

Show that the GLS is consistent if $u \in H^2(\Omega)$. [H3]

Question 4

Let a and b be real-valued bounded positive functions of a single argument satisfying the bounds $a \geq \alpha > \beta \geq b$ and let $f \in L^2(\Omega)$. Consider the nonlinear PDE

$$\begin{aligned} -\nabla\cdot(c(u)\nabla u) &= f, \qquad x \in \Omega, \\ u &= 0, \qquad x \in \partial\Omega, \end{aligned}$$

with $c = a - b$.

(a) Assume an approximation of the solution $u_k \in H_0^1(\Omega)$ to be known such that

$$-\nabla\cdot(c(u_k)\nabla u_k) = f_k.$$

Suggest a fixed point iteration which depends on c, u_k, and the residual $r_k = f_k - f$. [P5]

(b) A kind of 'split' fixed point iteration for this problem is the following one:

$$-\nabla\cdot(a(u_k)\nabla u_{k+1}) = f - \nabla\cdot(b(u_k)\nabla u_k), \tag{3}$$

and boundary conditions as before. Show that this iteration is consistent and derive a Galerkin-Picard FEM from this ansatz. Detail the linear equations to be solved in each iteration. [P5+P1]

(c) Produce an energy estimate for the iterate u_{k+1} using the FEM in (b). Show that there is in fact an energy estimate which is independent of the iteration index k. [P3]

(d) Show that (3) has a uniquely defined solution whenever a sufficiently regular u_k is given. [P2]

(e) Suppose that in the left side of (3), $a(u_k)$ is replaced with $a(u_{k+1})$. By a suitable linearisation, derive a Galerkin-Newton-Picard FEM, including the necessary linear equations to be solved in each iteration. [H2]

Good luck!
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