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Department of Information Technology
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Solutions to exam in Finite element methods II 2014-06-05

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Question 1

Note: a minus sign in front of the A is missing on the exam.

(a) Multiplying the PDE by a test-function v which vanishes at the boundary and integrating by parts, we get $(A\nabla u, \nabla v) + (b \cdot \nabla u, v) + (cu, v) = (f, v)$. Define $V = H_0^1(\Omega)$, the variational formulation reads: find $u \in V$ such that

$$a(u, v) =: (A\nabla u, \nabla v) + (b \cdot \nabla u, v) + (cu, v) = (f, v) := \ell(v), \quad \forall v \in V.$$

For the Galerkin approximation we consider a subspace $V_h \subset V$, find $u_h \in V_h$ such that

$$a(u_h, v) = \ell(v), \quad \forall v \in V_h.$$

(b) The assumptions are $(A\nabla u, \nabla u) \geq a_0(\nabla u, \nabla u)$, $c - \frac{1}{2}\nabla \cdot b > c_0 > 0$, and $f \in L^2(\Omega)$. Note also that

$$0 = (n \cdot bu^2, 1)_{\partial\Omega} = (\nabla \cdot (bu^2), 1)_\Omega = ((\nabla \cdot b)u, u)_\Omega + 2(b \cdot \nabla u, u)_\Omega. \tag{1}$$

We get

$$\begin{aligned} a(u, u) &= (A\nabla u, \nabla u) + (b \cdot \nabla u, u) + (cu, u) \geq a_0(\nabla u, \nabla u) + (cu, u) - \frac{1}{2}((\nabla \cdot b)u, u) \\ &\geq a_0(\nabla u, \nabla u) + c_0(u, u) = a_0\|\nabla u\|^2 + c_0\|u\|^2 \geq \min(a_0, c_0)\|u\|_V^2 = m\|u\|_V^2, \end{aligned} \tag{2}$$

for coercivity,

$$\begin{aligned} a(u, v) &= (A\nabla u, \nabla v) + (b \cdot \nabla u, v) + (cu, v) \\ &\leq \|A\|_{L^\infty(\Omega)}\|\nabla u\|\|\nabla v\| + \|b\|_{L^\infty(\Omega)}\|\nabla u\|\|v\| + \|c\|_{L^\infty(\Omega)}\|u\|\|v\| \\ &\leq C_{ab}\|\nabla u\|\|\nabla v\| + C_c\|u\|\|v\| \leq C\|u\|_V\|v\|_V \end{aligned} \tag{3}$$

for continuity, and

$$\ell(v) = (f, v) = \|f\|\|v\| \leq \|f\|\|v\|_V, \tag{4}$$

which together show that the Lax-Milgram lemma can be applied.

(c) We have the basic Galerkin orthogonality

$$a(u - u_h, v) = 0 \quad \forall v \in V_h. \tag{5}$$

We start by proving Cea's lemma and then use the interpolation estimate

$$\begin{aligned} \|u - u_h\|_V^2 &\leq \frac{1}{m}a(u - u_h, u - u_h) = \frac{C_a}{m}a(u - u_h, u - v) \\ &\leq \frac{C_a}{m}\|u - u_h\|_V\|u - v\|_V. \end{aligned} \tag{6}$$

We obtain for a Poincaré constant C_p,

$$\|u - u_h\|_V \leq \frac{C_a}{m}\|u - v\|_V \leq \frac{C_aC_p}{m}\|\nabla(u - \pi u)\| \leq \frac{C_aC_pC_\pi}{m}h^{\min(2,p)}|u|_{H^3(\Omega)}. \tag{7}$$

(d) First is to show that the solution the the variational problem minimize the functional $F(v) = \frac{1}{2}a(v,v) - \ell(v)$. We obtain

$$
\begin{aligned}
F(u+v) &= \frac{1}{2}a(u+v, u+v) - F(u+v) \\
&= \frac{1}{2}(a(u,u) + 2a(u,v) + a(v,v)) - \ell(u) - \ell(v) \\
&= F(u) + a(u,v) - \ell(u) + \frac{1}{2}a(v,v) \\
&\geq F(u).
\end{aligned}
\tag{8}
$$

Then we must show that a minimizer to $F(v)$ solves the variational problem. Let $g(\epsilon) = F(u + \epsilon v)$, we obtain

$$
g(\epsilon) = \frac{1}{2}a(u+\epsilon v, u+\epsilon v) - \ell(u+\epsilon v). \tag{9}
$$

Differentiating g with respect the ϵ, we have

$$
g'(\epsilon) = a(u,v) - \epsilon a(v,v) - \ell(v) \tag{10}
$$

and setting $\epsilon = 0$ we have the variational formulation. Note that $V_h \subset V$, we can concludes the proof.

Question 2

(a) We have the system

$$
e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix}
L_1(1) & L_1(r) & L_1(s) & L_1(t) & L_1(rs) & L_1(rt) & L_1(st) & L_1(rst) \\
L_2(1) & L_2(r) & L_2(s) & L_2(t) & L_2(rs) & L_2(rt) & L_2(st) & L_2(rst) \\
L_3(1) & L_3(r) & L_3(s) & L_3(t) & L_3(rs) & L_3(rt) & L_3(st) & L_3(rst) \\
L_4(1) & L_4(r) & L_4(s) & L_4(t) & L_4(rs) & L_4(rt) & L_4(st) & L_4(rst) \\
L_5(1) & L_5(r) & L_5(s) & L_5(t) & L_5(rs) & L_5(rt) & L_5(st) & L_5(rst) \\
L_6(1) & L_6(r) & L_6(s) & L_6(t) & L_6(rs) & L_6(rt) & L_6(st) & L_6(rst) \\
L_7(1) & L_7(r) & L_7(s) & L_7(t) & L_7(rs) & L_7(rt) & L_7(st) & L_7(rst) \\
L_8(1) & L_8(r) & L_8(s) & L_8(t) & L_8(rs) & L_8(rt) & L_8(st) & L_8(rst)
\end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} \tag{11}
$$

Which is

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} \tag{12}
$$

(b) We have

$$
\|u - u_h\|_h \leq \|u - v\|_h + \|v - u_h\|_h \tag{13}
$$

for $v \in V_h$. For the first term choose v as $\sup_{v \in V_h} \|u - v\|_h$. For the second term we have

$$
\begin{aligned}
\|v - u_h\|_h^2 \leq & \frac{1}{m}a_h(v - u_h, v - u_h) = \frac{1}{m}a_h(v - u + u - u_h, v - u_h) \\
&= \frac{1}{m}\left(a_h(v - u, v - u_h) + a_h(u - u_h, v - u_h)\right) \\
&= \frac{1}{m}\left(C_\alpha \|v - u\|_h \|v - u_h\|_h + a_h(u, v - u_h) - \ell(v - u_h)\right)
\end{aligned}
\tag{14}
$$

which implies that

$$\begin{aligned}\|v-u_h\|_h &\leq \frac{1}{m}\left(C_\alpha\|v-u\|_h + \frac{a_h(u,v-u_h)-\ell(v-u_h)}{\|v-u_h\|_h}\right)\\ &\leq \frac{1}{m}\left(C_\alpha\|v-u\|_h + \sup_{w\in V_h}\frac{|a_h(u,w)-\ell(w)|}{\|w\|_h}\right)\end{aligned} \tag{15}$$

Added the bound for the first and second term we have

$$\begin{aligned}\|u-u_h\|_h &\leq (1+\frac{C_\alpha}{m})\inf_{v\in V_h}\|u-v\|_h + \frac{1}{m}\sup_{w\in V_h}\frac{|a_h(u,w)-\ell(w)|}{\|w\|_h}\\ &\leq C\left(\inf_{v\in V_h}\|u-v\|_h + \sup_{w\in V_h}\frac{|a_h(u,w)-\ell(w)|}{\|w\|_h}\right)\end{aligned} \tag{16}$$

for $C = 1+\frac{C_\alpha}{m}$.

Question 3

(a) Lets try to bound the gradient of the solution in terms of data. We have the variational form: find $u \in V = H_0^1(\Omega)$

$$\epsilon(\nabla u, \nabla v) + (b\cdot\nabla u, v) = (f, v) \quad \forall v \in V. \tag{17}$$

We obtain

$$\epsilon\|\nabla u\|^2 = \epsilon(\nabla u,\nabla u) + (b\cdot\nabla u, u) = (f,v) \leq \|f\|\|v\| \leq C_{\mathrm{P-F}}\|f\|\|\nabla v\|. \tag{18}$$

where we used that

$$0 = (n\cdot bu^2, 1)_{\partial\Omega} = (\nabla\cdot(bu^2),1)_\Omega = ((\nabla\cdot b)u,u)_\Omega + 2(b\cdot\nabla u, u)_\Omega. \tag{19}$$

together with $\nabla\cdot b = 0$.

(b) First from the continuous problem we have that u solves: find $u \in V = H_0^1(\Omega)$ such that

$$\epsilon(\nabla u, \nabla v) + (b\cdot\nabla u, v) = (f, v) \quad \forall v \in V. \tag{20}$$

For the remaining part we have,

$$\sum_{T\in\mathcal{T}}(-\epsilon\Delta u + b\cdot\nabla u, -\epsilon\Delta v + b\cdot\nabla v)) = \sum_{T\in\mathcal{T}}(f, -\epsilon\Delta v + b\cdot\nabla v). \tag{21}$$

Also since, $u \in H^2(\Omega)\cap V$, we have that $-\epsilon\Delta : H^2(\Omega) \to L^2(\Omega)$, but $-\epsilon\Delta|_T : H^1(T) \not\to L^2(T)$ is not true in general, so we can not test for all $v \in V$. The elementwice operator $-\epsilon\Delta|_T : V_h \to L^2(T)$ is true since all $v \in V_h$ are polynomials of finite dimension. Multiplying the original equation with a test function in $w \in L^2(\Omega)$ defined as $w_T = -\epsilon\Delta v + b\cdot\nabla v \in L^2(T)$ on element T, we have that

$$\begin{aligned}&(\epsilon\Delta u, w) + (b\cdot\nabla u, w) = (f,w)\\ \Leftrightarrow &\sum_{T\in\mathcal{T}}(-\epsilon\Delta u + b\cdot\nabla u, w) = \sum_{T\in\mathcal{T}}(f,w),\end{aligned} \tag{22}$$

for all $v \in V_h$. Using the first and second part together we obtain that that $u \in H^2\cap V$ satisfies

$$a_h(u,v) = \ell(v) \quad \forall v \in V_h. \tag{23}$$

Question 4

(a) A typical Picard iteration for the problem is

$$-\nabla \cdot (c(u_k)\nabla u_{k+1}) = f.$$

With u_k as given we find by linearity that

$$-\nabla \cdot (c(u_k)\nabla(u_k - u_{k+1})) = r_k,$$

with homogeneous Dirichlet boundary conditions.

(b) The suggested iteration is clearly consistent since it is satisfied identically by any solution u to the PDE itself. Using the boundary conditions we readily find the variational formulation find $u_{k+1} \in H_0^1(\Omega)$ such that

$$(\nabla v, a(u_k)\nabla u_{k+1}) = (v, f) + (\nabla v, b(u_k)\nabla u_k)$$

for all $v \in H_0^1(\Omega)$. This implies the FEM

$$\begin{aligned} A_k\xi^{k+1} &= b + B_k\xi^k, \\ A_k(i,j) &= (\nabla\varphi_i, a(U_k)\nabla\varphi_j), \\ B_k(i,j) &= (\nabla\varphi_i, b(U_k)\nabla\varphi_j), \\ b_i &= (\varphi_i, f), \\ U_k &= \sum_j \xi_j^k \varphi_j. \end{aligned}$$

(c) By taking $v = u_{k+1}$ in the variational formulation we produce the energy estimate

$$\begin{aligned} \alpha\|\nabla u_{k+1}\|^2 &\le (\nabla u_{k+1}, a(u_k)\nabla u_{k+1}) = (u_{k+1}, f) + (\nabla u_{k+1}, b(u_k)\nabla u_k) \\ &\le \|u_{k+1}\|\|f\| + \beta\|\nabla u_{k+1}\|\|\nabla u_k\| \le C\|f\|\|\nabla u_{k+1}\| + \beta\|\nabla u_{k+1}\|\|\nabla u_k\|. \end{aligned}$$

Hence a suitable *a priori* bound is

$$\|\nabla u_{k+1}\| \le \alpha^{-1}C\|f\| + \alpha^{-1}\beta\|\nabla u_k\|,$$

which if $\kappa := \alpha^{-1}\beta < 1$ can be iterated to give

$$\begin{aligned} &\le \alpha^{-1}C\|f\| + \kappa\alpha^{-1}C\|f\| + \kappa^2\|\nabla u_{k-1}\| \le ... \\ &\le (1 + \kappa + ... + \kappa^k)\alpha^{-1}C\|f\| + \kappa^{k+1}\|\nabla u_0\| \\ &\le \frac{\alpha^{-1}C\|f\|}{1-\kappa} + \|\nabla u_0\|. \end{aligned}$$

(d) Here we show that the LM lemma is applicable, but the problem is symmetric so the RR theorem is also applicable. We have

$$\begin{aligned} |l(v)| &= |(v, f) + (\nabla v, b(u_k)\nabla u_k)| \le \\ &\le \max(\|f\|, \beta\|\nabla u_k\|)\,\|v\|_{H^1(\Omega)}. \\ |a(u,v)| &= |(\nabla v, a(u_k)\nabla u)| \le \\ &\le A_k\|\nabla v\|\|\nabla u\| \le A_k\|v\|_{H^1(\Omega)}\|u\|_{H^1(\Omega)}, \end{aligned}$$

where $a(u_k) \le A_k$ (recall that a is bounded). Finally,

$$\begin{aligned} a(u,u) &\ge \alpha\|\nabla u\|^2 \ge \frac{\alpha}{C^2}\|u\|^2, \\ \implies a(u,u) &\ge \frac{\alpha}{2}\min(1, 1/C^2)\|u\|^2_{H^1(\Omega)}. \end{aligned}$$

(e) We arrive at the Newton method by linearising around the previous iterate u_k. Put $u_{k+1} = u_k + \delta$. Then this is achieved by $a(u_{k+1})\nabla u_{k+1} = a(u_k+\delta)\nabla(u_k+\delta) = a(u_k)\nabla(u_k + \delta) + \delta a'(u_k)\nabla u_k + O(\delta^2)$. The variational formulation becomes find $u_{k+1} \in H_0^1(\Omega)$ such that

$$(\nabla v, a(u_k)\nabla u_{k+1}) + (\nabla v, u_{k+1}a'(u_k)\nabla u_k) = (v, f) + (\nabla v, b(u_k)\nabla u_k) + (\nabla v, u_k a'(u_k)\nabla u_k)$$

for all $v \in H_0^1(\Omega)$. The FEM becomes

$$(A_k + C_k)\xi^{k+1} = b + (B_k + C_k)\xi^k,$$

with matrices A and B as before and with

$$C_k(i,j) = (\nabla\varphi_i, \varphi_j a'(U_k)\nabla U_k).$$