Note: a minus sign in front of the $A$ is missing on the exam.

(a) Multiplying the PDE by a test-function $v$ which vanishes at the boundary and integrating by parts, we get $(Au, v) + (b \cdot \nabla u, v) + (cu, v) = (f, v)$. Define $V = H^1_0(\Omega)$, the variational formulation reads: find $u \in V$ such that

$$a(u, v) := (Au, v) + (b \cdot \nabla u, v) + (cu, v) = (f, v) := \ell(v), \quad \forall v \in V.$$ 

For the Galerkin approximation we consider a subspace $V_h \subset V$, find $u_h \in V_h$ such that

$$a(u_h, v) = \ell(v), \quad \forall v \in V_h.$$ 

(b) The assumptions are $(Au, u) \geq a_0(\nabla u, \nabla u)$, $c - \frac{1}{2} \nabla \cdot b > c_0 > 0$, and $f \in L^2(\Omega)$. Note also that

$$0 = (u \cdot (u^2), 1)_{\partial \Omega} = (\nabla \cdot (bu^2), 1)_{\Omega} = ((\nabla \cdot b)u, u)_{\Omega} + 2(b \cdot \nabla u, u)_{\Omega}. \quad (1)$$ 

We get

$$a(u, v) = (Au, v) + (b \cdot \nabla u, v) + (cu, v) \geq a_0(\nabla u, \nabla u) + (cu, v) - \frac{1}{2}((\nabla \cdot b)u, u) \geq a_0(\nabla u, \nabla u) + c_0(u, u) = a_0 \|\nabla u\|^2 + c_0 \|u\|^2 \geq \min(a_0, c_0) \|u\|^2 = m \|u\|^2, \quad (2)$$

for coercivity,

$$a(u, v) = (Au, v) + (b \cdot \nabla u, v) + (cu, v) \leq \|A\|_{L^\infty(\Omega)} \|\nabla u\| \|v\| + \|b\|_{L^\infty(\Omega)} \|\nabla u\| \|v\| + \|c\|_{L^\infty(\Omega)} \|u\| \|v\| \leq C_{ab} \|\nabla u\| \|v\| + C_c \|u\| \|v\| \leq C \|u\| \|v\|, \quad (3)$$

for continuity, and

$$\ell(v) = (f, v) = \|f\| \|v\| \leq \|f\| \|v\|, \quad (4)$$

which together show that the Lax-Milgram lemma can be applied.

(c) We have the basic Galerkin orthogonality

$$a(u - u_h, v) = 0, \quad \forall v \in V_h. \quad (5)$$

We start by proving Cea’s lemma and then use the interpolation estimate

$$\|u - u_h\|_V^2 \leq \frac{1}{m} a(u - u_h, u - u_h) = \frac{C_a}{m} a(u - u_h, u - v) \leq \frac{C_a}{m} \|u - u_h\|_V \|u - v\|_V. \quad (6)$$

We obtain for a Poincaré constant $C_p$,

$$\|u - u_h\|_V \leq \frac{C_a}{m} \|u - v\|_V \leq \frac{C_a C_p}{m} \|\nabla (u - \pi u)\| \leq \frac{C_a C_p C_{\pi}}{m} h_{\min(2,p)} |u|_{H^3(\Omega)}. \quad (7)$$
(d) First is to show that the solution the the variational problem minimize the functional $F(v) = \frac{1}{2}a(v, v) - \ell(v)$. We obtain

$$F(u + v) = \frac{1}{2}a(u + v, u + v) - F(u + v)$$

$$= \frac{1}{2}(a(u, u) + 2a(u, v) + a(v, v)) - \ell(u) - \ell(v)$$

$$= F(u) + a(u, v) - \ell(u) + \frac{1}{2}a(v, v)$$

$$\geq F(u).$$

Then we must show that a minimizer to $F(v)$ solves the variational problem. Let $g(\epsilon) = F(u + \epsilon v)$, we obtain

$$g(\epsilon) = \frac{1}{2}a(u + \epsilon v, u + \epsilon v) - \ell(u + \epsilon v).$$

Differentiating $g$ with respect the $\epsilon$, we have

$$g'(\epsilon) = a(u, v) - \epsilon a(v, v) - \ell(v)$$

and setting $\epsilon = 0$ we have the variational formulation. Note that $V_h \subset V$, we can concludes the proof.

**Question 2**

(a) We have the system

$$e_1 = \begin{pmatrix} L_1(1) & L_1(r) & L_1(s) & L_1(t) & L_1(rst) \\ L_2(1) & L_2(r) & L_2(s) & L_2(t) & L_2(rst) \\ L_3(1) & L_3(r) & L_3(s) & L_3(t) & L_3(rst) \\ L_4(1) & L_4(r) & L_4(s) & L_4(t) & L_4(rst) \\ L_5(1) & L_5(r) & L_5(s) & L_5(t) & L_5(rst) \\ L_6(1) & L_6(r) & L_6(s) & L_6(t) & L_6(rst) \\ L_7(1) & L_7(r) & L_7(s) & L_7(t) & L_7(rst) \\ L_8(1) & L_8(r) & L_8(s) & L_8(t) & L_8(rst) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

Which is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}$$

(11)

(b) We have

$$\|u - u_h\|_h \leq \|u - v\|_h + \|v - u_h\|_h$$

for $v \in V_h$. For the first term choose $v$ as $\sup_{v \in V_h} \|u - v\|_h$. For the second term we have

$$\|v - u_h\|_h^2 \leq \frac{1}{m}a_h(v - u_h, v - u_h) = \frac{1}{m}a_h(v - u + u - u_h, v - u_h)$$

$$= \frac{1}{m} (a_h(v - u, v - u_h) + a_h(u - u_h, v - u_h))$$

$$= \frac{1}{m} (C_{\alpha} \|v - u\|_h \|v - u_h\|_h + a_h(u, v - u_h) - \ell(v - u_h))$$

(14)
which implies that
\[
\|v - u_h\|_h \leq \frac{1}{m} \left( C_\alpha \|v - u\|_h + \frac{a_h(u, v - u_h) - \ell(v - u_h)}{\|v - u_h\|_h} \right)
\]
\[
\leq \frac{1}{m} \left( C_\alpha \|v - u\|_h + \sup_{w \in V_h} \frac{|a_h(u, w) - \ell(w)|}{\|w\|_h} \right)
\]
(15)

Added the bound for the first and second term we have
\[
\|u - u_h\|_h \leq (1 + \frac{C_\alpha}{m}) \inf_{v \in V_h} \|u - v\|_h + \frac{1}{m} \sup_{w \in V_h} \frac{|a_h(u, w) - \ell(w)|}{\|w\|_h}
\]
\[
\leq C \left( \inf_{v \in V_h} \|u - v\|_h + \sup_{w \in V_h} \frac{|a_h(u, w) - \ell(w)|}{\|w\|_h} \right)
\]
(16)

for $C = 1 + \frac{C_\alpha}{m}$.

**Question 3**

(a) Let’s try to bound the gradient of the solution in terms of data. We have the variational form: find $u \in V = H^1_0(\Omega)$
\[
\epsilon(\nabla u, \nabla v) + (b \cdot \nabla u, v) = (f, v) \quad \forall v \in V.
\]
(17)

We obtain
\[
\epsilon \|\nabla u\|^2 = \epsilon(\nabla u, \nabla u) + (b \cdot \nabla u, u) = (f, v) \leq \|f\| \|v\| \leq C_{P-\ell} \|f\| \|\nabla v\|.
\]
(18)

where we used that
\[
0 = (n \cdot bu^2, 1)_{\partial \Omega} = (\nabla \cdot (bu^2), 1)_{\Omega} = ((\nabla \cdot b)u, u)_{\Omega} + 2(b \cdot \nabla u, u)_{\Omega}.
\]
(19)

together with $\nabla \cdot b = 0$.

(b) First from the continuous problem we have that $u$ solves: find $u \in V = H^1_0(\Omega)$ such that
\[
\epsilon(\nabla u, \nabla v) + (b \cdot \nabla u, v) = (f, v) \quad \forall v \in V.
\]
(20)

For the remaining part we have,
\[
\sum_{T \in \mathcal{T}} (-\epsilon \Delta u + b \cdot \nabla u, -\epsilon \Delta v + b \cdot \nabla v) = \sum_{T \in \mathcal{T}} (f, -\epsilon \Delta v + b \cdot \nabla v).
\]
(21)

Also since, $u \in H^2(\Omega) \cap V$, we have that $-\epsilon \Delta : H^2(\Omega) \to L^2(\Omega)$, but $-\epsilon \Delta |_T : H^1(T) \not\to L^2(T)$ is not true in general, so we can not test for all $v \in V$. The elementwise operator $-\epsilon \Delta |_T : V_h \to L^2(T)$ is true since all $v \in V_h$ are polynomials of finite dimension. Multiplying the original equation with a test function in $w \in L^2(\Omega)$ defined as $w_T = -\epsilon \Delta v + b \cdot \nabla v \in L^2(T)$ on element $T$, we have that
\[
(\epsilon \Delta u, w) + (b \cdot \nabla u, w) = (f, w)
\]
\[
\Leftrightarrow \sum_{T \in \mathcal{T}} (-\epsilon \Delta u + b \cdot \nabla u, w) = \sum_{T \in \mathcal{T}} (f, w),
\]
(22)

for all $v \in V_h$. Using the first and second part together we obtain that that $u \in H^2 \cap V$ satisfies
\[
a_h(u, v) = \ell(v) \quad \forall v \in V_h.
\]
(23)
Question 4

(a) A typical Picard iteration for the problem is
\[ -\nabla \cdot (c(u_k)\nabla u_{k+1}) = f. \]

With \( u_k \) as given we find by linearity that
\[ -\nabla \cdot (c(u_k)\nabla (u_k - u_{k+1})) = r_k, \]
with homogeneous Dirichlet boundary conditions.

(b) The suggested iteration is clearly consistent since it is satisfied identically by any solution \( u \) to the PDE itself. Using the boundary conditions we readily find the variational formulation find \( u_{k+1} \in H^1_0(\Omega) \) such that
\[ (\nabla v, a(u_k)\nabla u_{k+1}) = (v, f) + (\nabla v, b(u_k)\nabla u_k) \]
for all \( v \in H^1_0(\Omega) \). This implies the FEM
\[
\begin{align*}
A_k\xi_{k+1} &= b + B_k\xi_k, \\
A_k(i, j) &= (\nabla \varphi_i, a(U_k)\nabla \varphi_j), \\
B_k(i, j) &= (\nabla \varphi_i, b(U_k)\nabla \varphi_j), \\
b_i &= (\varphi_i, f), \\
U_k &= \sum_j \xi_k \varphi_j.
\end{align*}
\]

(c) By taking \( v = u_{k+1} \) in the variational formulation we produce the energy estimate
\[
a\|\nabla u_{k+1}\|^2 \leq (\nabla u_{k+1}, a(u_k)\nabla u_{k+1}) = (u_{k+1}, f) + (\nabla u_{k+1}, b(u_k)\nabla u_k) \\
\leq \|u_{k+1}\|\|f\| + \beta\|\nabla u_{k+1}\|\|\nabla u_k\| \leq C\|f\|\|\nabla u_{k+1}\| + \beta\|\nabla u_{k+1}\|\|\nabla u_k\|.
\]

Hence a suitable \( a \) priori bound is
\[
\|\nabla u_{k+1}\| \leq \alpha^{-1}C\|f\| + \alpha^{-1}\beta\|\nabla u_k\|,
\]
which if \( \kappa := \alpha^{-1}\beta < 1 \) can be iterated to give
\[
\begin{align*}
&\leq \alpha^{-1}C\|f\| + \kappa\alpha^{-1}C\|f\| + \kappa^2\|\nabla u_{k-1}\| \leq \ldots \\
&\leq (1 + \kappa + \ldots + \kappa^k)\alpha^{-1}C\|f\| + \kappa^{k+1}\|\nabla u_0\| \\
&\leq \frac{\alpha^{-1}C\|f\|}{1 - \kappa} + \|\nabla u_0\|.
\end{align*}
\]

(d) Here we show that the LM lemma is applicable, but the problem is symmetric so the RR theorem is also applicable. We have
\[
|l(v)| = |(v, f) + (\nabla v, b(u_k)\nabla u_k)| \leq \max(|f|, \beta\|\nabla u_k\|)\|v\|_{H^1(\Omega)},
\]
\[
|a(u, v)| = |(\nabla v, a(u_k)\nabla u)| \leq A_k\|\nabla v\|\|\nabla u\| \leq A_k\|v\|_{H^1(\Omega)}\|u\|_{H^1(\Omega)},
\]
where \( a(u_k) \leq A_k \) (recall that \( a \) is bounded). Finally,
\[
\begin{align*}
a(u, u) &\geq a\|\nabla u\|^2 \geq \frac{\alpha}{C^2}\|u\|^2, \\
\Rightarrow a(u, u) &\geq \frac{\alpha}{2} \min(1, 1/C^2)\|u\|^2_{H^1(\Omega)}.
\end{align*}
\]
We arrive at the Newton method by linearising around the previous iterate $u_k$. Put $u_{k+1} = u_k + \delta$. Then this is achieved by $a(u_{k+1}) \nabla u_{k+1} = a(u_k + \delta) \nabla (u_k + \delta) = a(u_k) \nabla (u_k + \delta) + \delta a'(u_k) \nabla (u_k) + O(\delta^2)$. The variational formulation becomes find $u_{k+1} \in H^1_0(\Omega)$ such that

$$(\nabla v, a(u_k) \nabla u_{k+1}) + (\nabla v, u_{k+1} a'(u_k) \nabla u_k) = (v, f) + (\nabla v, b(u_k) \nabla u_k) + (\nabla v, u_k a'(u_k) \nabla u_k)$$

for all $v \in H^1_0(\Omega)$. The FEM becomes

$$(A_k + C_k) \xi^{k+1} = b + (B_k + C_k) \xi^k,$$

with matrices $A$ and $B$ as before and with

$$C_k(i, j) = (\nabla \varphi_i, \varphi_j a'(U_k) \nabla U_k).$$