Linear Elasticity

Project 1

This project is closely related to Chapter 11 in *The finite element method: theory, implementation, and applications* by Larson and Bengzon. Read Chapter 11 carefully and take advantage of the analysis and implementation done there.

The linear elastic problem for static equilibrium of a homogeneous isotropic body $\Omega \subset \mathbb{R}^2$ under the assumption of small deformations and strains reads: find the symmetric stress tensor $\boldsymbol{\sigma} = [\sigma_{ij}]_1^2$ and the displacement vector $\boldsymbol{u} = [u_i]_1^2$, such that

$$-\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{f}, \qquad \text{in } \Omega \tag{1a}$$

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I}, \qquad \text{in } \Omega \tag{1b}$$

$$\boldsymbol{u} = \boldsymbol{0}, \qquad \text{on } \Gamma_D \tag{1c}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{g}, \qquad \text{on } \Gamma_N \tag{1d}$$

Here, $\boldsymbol{f}$ is a given body force, and $\boldsymbol{g}$ a given traction load acting along a segment Γ_N of the boundary, which has outward unit normal $\boldsymbol{n}$. Along the rest of the boundary Γ_D the body is clamped and can not be displaced. The elastic properties of the body are governed by the positive constants λ and μ called the Lamé parameters. We imagine Ω to be the cross section of a long slender structure aligned along the x_3-axis. For such structures a state of plane strain is applicable, which essentially means that all loads and are confined to the x_1x_2-plane and that no quantities depend on x_3. Further, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = [\varepsilon_{ij}]_1^2$ is the strain tensor with components

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,2 \tag{2}$$

The divergence of the 2×2 tensor $\boldsymbol{\sigma}$ and the 2×1 vector $\boldsymbol{u}$ is defined by

$$\nabla \cdot \boldsymbol{\sigma} = \left[\sum_{j=1}^{2} \frac{\partial \sigma_{ij}}{\partial x_j} \right]_{i=1}^{2}, \qquad \nabla \cdot \boldsymbol{u} = \sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i} \tag{3}$$

Finally, $\boldsymbol{I}$ is the 2×2 identity matrix.

Problem 1. Derive the variational formulation of (1). Present the result on the form: find $\boldsymbol{u} \in \mathcal{V} = \{\boldsymbol{v} \in [H^1(\Omega)]^2 : \boldsymbol{v}|_{\Gamma_D} = \boldsymbol{0}\}$, such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = l(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathcal{V} \tag{4}$$

and define $a(\cdot, \cdot)$ and $l(\cdot)$.

Problem 2. Verify that the conditions for the Lax-Milgram lemma are satisfied for the variational equation (4). For simplicity, you only have to consider the case of homogeneous Dirichlet boundary conditions $\boldsymbol{u} = \boldsymbol{0}$ on the whole boundary $\partial\Omega$. *Hint:* Korn's inequality is useful.

Problem 3. Derive the finite element approximation to equation (4) by introducing a discrete space of continuous piecewise linear vector polynomials on a triangulation $\mathcal{K}$. Find a basis for the discrete space using the usual hat functions.

Problem 4. Implement the finite element method in Matlab. You can use the following m-files as a starting point.

```
function Ke = stiffness(x,y,mu,lambda)
area=polyarea(x,y); % x and y are 3 x 1 and hold node coordinates
b=[y(2)-y(3); y(3)-y(1); y(1)-y(2)];
c=[x(3)-x(2); x(1)-x(3); x(2)-x(1)];
D=mu*[2 0 0; 0 2 0; 0 0 1] + lambda*[1 1 0; 1 1 0; 0 0 0];
Be=[b(1)   0  b(2)   0  b(3)   0 ;
      0  c(1)   0  c(2)   0  c(3);
    c(1) b(1) c(2) b(2) c(3) b(3)]/2/area;
Ke=Be'*D*Be*area;
```

```
function Fe = load(x,y)
```

```
area=polyarea(x,y);
f=force(mean(x),mean(y));
Fe=(f(1)*[1 0 1 0 1 0]'+f(2)*[0 1 0 1 0 1]')*area/3;


function [K,F] = assemble(p,e,t)
ndof=2*size(p,2);
K=sparse(ndof,ndof);
F=zeros(ndof,1);
dofs=zeros(6,1);
E=1;  nu=0.3;
lambda=E*nu/((1+nu)*(1-2*nu));  mu=E/(2*(1+nu));
for i=1:size(t,2)
  nodes=t(1:3,i);
  x=p(1,nodes);  y=p(2,nodes);
  dofs(1:2:end)=2*nodes-1;  dofs(2:2:end)=2*nodes;
  Ke=stiffness(x,y,mu,lambda);
  Fe=load(x,y);
  K(dofs,dofs)=K(dofs,dofs)+Ke;
  F(dofs)=F(dofs)+Fe;
end
```

Problem 5. Now let $\Omega = [0,1]^2$ with clamped boundary. Assume $E = 1$, $\nu = 0.3$, and body force

$$\boldsymbol{f} = \begin{bmatrix} (\lambda+\mu)(1-2x)(1-2y) \\ -2\mu y(1-y) - 2(\lambda+2\mu)x(1-x) \end{bmatrix}$$

The analytical solution is given by $\boldsymbol{u} = [0,\ -x(1-x)y(1-y)]$. Plot the displacement components. Validate your code by computing the energy norm $a(\boldsymbol{u}_h, \boldsymbol{u}_h)$. It should converge to $(\lambda + 3\mu)/90$.

Problem 6. Now try different data sets and evaluate your solver by comparing the solution to the Linear Elasticity solver in COMSOL Multiphysics. Very boundary data, load forcing, and parameters.

Problem 7. Modal analysis is a crucial part of linear elasticity. Let K be the stiffness matrix as derived in Problem 3 and let M be the mass matrix, see

MGL Chapter 11 pp 269-270. Implement and solve the generalized eigenvalue problem,

$$\boldsymbol{K\phi} = \omega^2 \boldsymbol{M\phi}, \tag{5}$$

for the eigenvectors ϕ and eigenvalues $\lambda = \omega^2$. The Matlab function `eigs` can be used to compute the lowest eigenvalues (which are the crucial ones).

Problem 8. A mesh of the famous L-shaped domain is obtained by typing `[p,e,t]=initmesh('lshapeg')`. Compute and plot the ten lowest eigenmodes on this domain. Assume elastic constants $\rho = 1$, $E = 1$, and $\nu = 0.3$. Evaluate your result using COMSOL Multiphysics.

Problem 9. Construct a problem of your own which demonstrates how modal analysis is critical in structural mechanics. Use COMSOL Multiphysics and study dynamic linear elasticity with a carefully chosen forcing function.

Problem 10. Feel free to further investigate your own code and/or COMSOL Multiphysics doing more experiments and investigations.